Abstract. In this article we study some parabolas naturally associated with a generic convex quadrangle. It is shown that the quadrangle defines, through these parabolas, a one-parameter-family of quadrangles, containing the quadrangle of reference. The members of this family share the same diagonal lines, Newton line and ratio of diagonals. In addition it is shown that this family contains a unique cyclic quadrangle.

1. Introduction

The aim in this article is to show that a given quadrangle $q = ABCD$ can be considered as a member of a one-parameter-family of quadrangles sharing the same diagonals and ratio of diagonals. In addition it is shown that the family contains exactly one cyclic member-quadrangle. The quadrangles of the family have their pairs of opposite sides tangent to two parabolas $\{\kappa_1, \kappa_2\}$, which can be defined directly from the quadrangle but have also another aspect, as envelopes of diagonals of inscribed in $q$ parallelograms. Section 2 discusses in short these parallelograms, since their properties lead to important for the sequel relations between the two parabolas. In section 3 we define the two parabolas as envelopes of certain lines and discuss in some detail their mutual relationship, the discussion culminating with theorem 3.5. In section 4 we define the two parabolas as envelopes of certain lines and discuss in some detail their mutual relationship, the discussion culminating with theorem 3.5. In section 4 we discuss the structures involving the directrices of the two parabolas. This is useful for the determination of the particular cyclic member of the one-parameter-family. In section 5 we detect and study a third parabola, the tangents of which parametrize the members of the family. Finally in section 6 we combine all the previous results to our main theorems 6.1 and 6.2.

2. Special inscribed parallelograms

In a generic convex quadrangle $q = ABCD$ we can easily inscribe infinite many parallelograms $q_t = A_tB_tC_tD_t$ with sides parallel to the diagonals $\{AC, BD\}$ of $q$ (See Figure 1).
Figure 1). The vertex $A_t$ of such a parallelogram can be selected on an arbitrary side, $AB$ say. Then, drawing a parallel to the diagonal $AC$, we define its intersection point $B_t$ with $BC$. Drawing further from $B_t$ a parallel to $BD$, we define its intersection $C_t$ with $CD$ and continuing this way, we obtain, using Thales theorem, an inscribed parallelogram $q_t = A_tB_tC_tD_t$ with sides parallel to the diagonals of $q$. An obvious property of these parallelograms is, that their centers $O_t$ move on the Newton line $IJ$ of $q$, joining the middles $\{I,J\}$ of the diagonals. The proof can be immediately read from figure 2. Next lemma shows that this is a characteristic property of the parallelograms of this kind.

Lemma 2.1. A parallelogram $q' = A'B'C'D'$, inscribed in the quadrangle $q = ABCD$, has its sides parallel to the diagonals of $q$, if and only if its center is on the Newton line of $q$.

Proof. The “if” part is the preceding remark. The “only if” part follows from the symmetry of the parallelogram w.r. to its center. By this, the parallelogram $A'B'C'D'$ can be considered to be defined by the intersections of corresponding sides of $q$ and its symmetric $q'' = A''B''C''D''$ w.r. to the center $O$ of $q'$ (See Figure 3). It is then easy to see
that $C''$ moves on a parallel $\nu'$ of the Newton line $\nu$ of $q$ and the triangles $\{FA'D'\}$ have constant angles as $O$ varies on $\nu$. This implies that, for all these points $O$ on the Newton line $\nu$ of $q$, the sides $A'D'$ have a fixed direction, which coincides with $BD$, when $O$ takes the position of $J$, thereby proving that $A'D'$ is always parallel to $BD$. Analogously is seen that $A'B'$ is parallel to the other diagonal $AC$ of $q$. □

Next theorem shows that this class of parallelograms is unique from another aspect.

**Theorem 2.2.** Given are a convex quadrangle $q = ABCD$ and two directions $\{\alpha, \beta\}$. For a point $X_0 \in AB$ we draw successively parallels to these directions to obtain respective points on the sides $X_1 \in BC$, $X_2 \in CD$, $X_3 \in DA$. If there are two points $\{X_0 \in AB\}$ such that $X_0X_1X_2X_3$ is a parallelogram, then the two directions are those of the diagonals of $q$.

**Proof.** In fact, consider the additional point $X_4 \in AB$, defined by the intersection of $AB$ with the parallel to $\beta$ from $X_3$ and the intersection point $Z = X_3X_4 \cap X_0X_1$ (See Figure 4). Introducing affine coordinates $\{X_0(x), X_4(y)\}$ along the line $AB$, we see easily (\cite{7}), that they are related linearly $y = ax + b$, that the triangle $\tau_x = X_0X_4Z$ remains similar to itself for all positions of $X_0$ and its vertex $Z$ moves on a fixed line $\varepsilon$. This line passes through the intersection $C_1$ of line $AD$ with the parallel to $\alpha$ from $C$ and through the intersection $D_1$ of line $BC$ with the parallel to $\beta$ from $D$.

The condition formulated in the statement of the theorem is equivalent with the coincidence of the two lines $\varepsilon$ and $AB$. But this is easily seen to be equivalent with $C_1 \equiv A$ and $D_1 \equiv B$, which proves the claim. □

**Corollary 2.3.** Under the notation and conventions of the previous theorem, all the quadrangles $X_0X_1X_2X_3$ are parallelograms, if and only if, the directions $\{\alpha, \beta\}$ are those of the diagonals of $q = ABCD$.

3. **Parabolas enveloping the diagonals**

The diagonals of the family of the parallelograms of the previous section envelope two parabolas. This follows from a fundamental property of the parabola, according to which, a linear relation between two lines defines a parabola tangent to these lines. This, in turn, is a special case of the general Chasles-Steiner generation method of conics through homographic relations between the points of two lines (\cite[p. 6]{4}). For the diagonal $B'D'$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Parallelograms with sides parallel to arbitrary directions $\{\alpha, \beta\}$}
\end{figure}
Figure 5. Parabola $\kappa_1$ tangent to sides $\{AD, BC\}$ and the diagonals

this is seen by introducing coordinates along the opposite sides $\{AD, BC\}$ (See Figure 5). Denoting the coordinates by \{B(b), C(c), B'(x)\} and \{A(a), D(d), D'(y)\}, and using the parallelity of the sides to the diagonals of $q$, we find that

\[ y = \frac{(a - d)x - (ab - cd)}{c - b}. \]

The linearity of the relation implies that the conic is a parabola $\kappa_1$. This, because for $x$ tending to infinity, the same happens to $y$, consequently the tangent line $B'(x)D'(y)$ tends to the line at infinity. Thus, the line at infinity is tangent to the conic, which therefore is a parabola ([2, p. 137]). The parabola is identified with the one tangent to the four lines $\{BC, AD, AC, BD\}$, having the well known properties of these kind of conics ([6, p. 54], [3, p. 178], [10]). Among other things, the circles $(E_1B'D')$ pass through the focus $F_1$ of the parabola and the point of contact $T_1$ of the tangent $B'D'$ is the harmonic conjugate of the intersection $G_1 = (B'D', \nu_1)$, where $\nu_1 = C'D'$ is the chord of contact points of the lines $\{BC, AD\}$ with the conic. It is also known that the symmedian from $E_1$ of the triangle $E_1C'D'$ passes through the focus $F_1$. Another feature of this configurations is also that the triangle $E_1E_2S_1$, with $S_1 = (EE_1, C'D')$ is self-polar relative to $\kappa_1$. Regarding the diagonal $B'D'$, we can thus formulate a property of it related to the circle through the focus.

**Lemma 3.1.** The diagonals $B'D'$ of the family of inscribed in $ABCD$ parallelograms $A'B'C'D'$, with sides parallel to the diagonals, are the chords of intersection points of the pencil of circles $\mathcal{P}_{E_1,F_1}$ passing through $\{E_1, F_1\}$, with the sides of the angle $BE_1A$.

**Lemma 3.2.** The Newton line of the quadrilateral $ABCD$ is tangent to the parabola at the middle $N_1$ of the segment $I_1J_1$ intercepted by the sides $\{AD, BC\}$. It is also parallel to the polar $C'D'$ of $E_1$ at half the distance of $E_1$ from that polar. Further, the ratio of the opposite sides $AD/BC = E_1J_1/E_1I_1$.

**Proof.** Consider the polar $C'D'$ of $E_1$ and its parallel $\nu$ at half the distance of $E_1$ from $C'D'$ (See Figure 6). By the well known properties of parabolas, line $\nu$ is tangent to the
Figure 6. The Newton line tangent to the parabola

parabola at its middle \( N_1 \). By one of the main properties of parabolas, the ratio of the segments intercepted on a variable tangent by three other fixed tangents to the parabola is constant ([4, p. 52]). Consequently all tangents \( A_1B_1 \) to the parabola are intersected by the three tangents \( \{AD, \nu, BC\} \) in segments \( \{A_1M_1, M_1B_1\} \) having constant ratio \( A_1M_1/M_1B_1 \). But this ratio is 1, as is seen by drawing \( A_1B_1 \) in the special position \( A_1B_1 = E_1C' \). A similar argument, for the diagonals as tangent lines, shows that \( \nu \) passes through the middles of the diagonals \( \{AC, BD\} \) and identifies the line \( \nu \) with the Newton line and points \( \{I_1, J_1\} \) with the respective middles of \( \{E_1C', E_1D'\} \). \[\square\]

Notice that, by an elementary property of the parabola, the projection of the focus on the tangents lies on the tangent at the vertex of the parabola, which thus is the Wallace-Simson line of the focus, with respect to any triangle circumscribed to the parabola ([1, p. 22]). Thus, the tangent \( A_1B_1 \) at the vertex of the parabola (See Figure 6) is the Wallace-Simson line of \( F_1 \) w.r. to the triangle \( E_1I_1J_1 \). By well known properties of parabolas, we have also the next corollary.

Corollary 3.3. Under the previous notation and conventions, the line \( E_1N_1 \) is parallel to the axis of the parabola, so that for every point on \( E_1N_1 \) the corresponding polar is parallel to the Newton line \( \nu \).

Let now \( \{G, H\} \) be the middles of the diagonals on the Newton line, \( \{P_1, Q_1\} \) be the contact points of the parabola with the diagonals and \( N_2 = (I_1J_1, P_1Q_1) \). Further, let the opposite sides \( \{AB, CD\} \) of the quadrangle intersect the Newton line correspondingly at points \( \{I_2, J_2\} \). The following lemma establishes a relation between the points \( \{N_1, N_2\} \) (See Figure 7).

Lemma 3.4. The points \( \{G, H\} \) are harmonic conjugate to \( \{N_1, N_2\} \) and point \( N_2 \) is the middle of the segment \( I_2J_2 \).

Proof. For \( S_1 = (EE_1, C'D') \), consider the triangle \( E_1E_2S_1 \), which is self-polar w.r. to the parabola. Hence for all points \( E \) on \( E_1S_1 \) the polar of \( E \) will pass through \( E_2 \). This is true in particular for \( P_1Q_1 \), so that \( \{E_2, P_1, Q_1, N_2\} \) are on the polar of \( E \). Thus \( N_2 \) is on the polars of points \( \{N_1, E\} \), hence its polar will be line \( N_1E \), so that \( (GHN_1N_2) = -1 \).
To show the second claim consider the lines \( \{ P_1D', C'Q_1 \} \) and see that they intersect at a point \( U_1 \) on line \( CD \). In fact, if \( V_1 = (P_1Q_1, EE_1) \), the two quadruples of collinear points \( (P_1Q_1V_1E_2) = (D'C'S_1E_2) = -1 \) are harmonic, hence, by well known theorem ([5, p. 90]) the lines \( \{ P_1D', Q_1C', S_1V_1 \} \) are concurrent and the lines \( \{ P_1C'', Q_1D', V_1S_1 \} \) are also concurrent at points \( \{ U_1, W_1 \} \) of the polar of \( E_2 \). In addition the triangle \( E_2U_1W_1 \) is self-polar w.r. to the parabola. But \( P_1D' \) is the polar of \( A \) and \( C'Q_1 \) is the polar of \( B \). Hence, \( AB \) is the polar of of \( U_1 \), consequently \( W_1 \in AB \). Analogously is seen that \( U_1 \) on \( CD \). The previous arguments imply that the pencil \( E_2(CBP_1C') \) is harmonic, and since \( E_2C' \) is parallel to \( I_1J_1 \) the other rays of the pencil intersect \( I_1J_1 \) at \( \{ I_2, J_2 \} \) and the middle \( N_2 \) of \( I_2J_2 \). \( \Box \)

Doing the same work for the second diagonal of the inscribed parallelograms, we find their envelope, which is a second parabola \( \kappa_2 \), tangent to the lines \( \{ AB, CD, AC, BD \} \). Figure 8 displays the resulting configuration of the two parabolas, denoting with indexed letters points having the same meaning relative to the parabola with the corresponding index. Thus, \( \{ P_2, Q_2 \} \) are the contact points of \( \kappa_2 \) with the diagonals, \( N_2 \) is its contact point with the Newton line etc.

**Theorem 3.5.** Under the notation and conventions adopted so far, the following are valid properties.

1. The triples \( \{(P_1, Q_1, N_2), (P_2, Q_2, N_1)\} \) of contact points of the two parabolas consist of collinear points.
2. The intersection \( E' = (P_1Q_1, P_2Q_2) \) of the previous lines defines the lines \( \{ E_1E', E_2E' \} \), which are parallel to the axes of the parabolas \( \{ \kappa_1, \kappa_2 \} \), hence these lines represent conjugate directions w.r. to both parabolas. The middles of all chords of \( \kappa_1 \) (resp. \( \kappa_2 \)) parallel to \( \nu \) lie on \( E_1E' \) (resp. \( E_2E' \)).
3. The polar \( \varepsilon \) of \( E' = (P_1Q_1, P_2Q_2) \) w.r. to either of the parabolas is the parallel to the Newton line \( \nu \) through \( E \).
(4) The line $\varepsilon'$, which is parallel to the Newton line through $E'$, is the polar of $R_1 = (P_1Q_1, \varepsilon)$ w.r. to $\kappa_2$ and also the polar of $R_2 = (P_2Q_2, \varepsilon)$ w.r. to $\kappa_1$.

(5) The line $\eta = EE'$ passes through the middle $M$ of $GH$ and is the polar of $R_1$ w.r. to $\kappa_1$ and also the polar of $R_2$ w.r. to $\kappa_2$. Points $\{N_1, N_2\}$ are respectively the middles of the segments $\{E'R_2, E'R_1\}$.

(6) $EGE'H$ is a parallelogram and $\{G, H\}$ are respectively the middles of $\{P_1P_2, Q_1Q_2\}$.

(7) It is $P_1C = AP_2$ and $BQ_2 = Q_1D$.

(8) The polars of points $P \in \varepsilon$ w.r. to the two parabolas pass all through $E'$ and there are two points $\{P_H, P_G\}$ on $\varepsilon$, lying symmetric w.r. to $E$, for which the two polars coincide. These are $P_H = (\varepsilon, E'H)$, $P_G = (\varepsilon, E'G)$ with respective polars $GE'$ and $HE'$.

(9) The triangle $P_GP_HE'$ is anticomplementary of the common, tangential to the two parabolas, triangle $EGH$ and is self-polar w.r. to both parabolas.

Proof. Nr-1 follows from lemma 3.4, since this implies that $N_2$ is the middle of segment $I_2J_2$ intercepted on the Newton line by the opposite sides $\{AB, CD\}$ of the quadrangle, which, by the analogous to lemma 3.2 for $\kappa_2$, is the contact point of $\kappa_2$ with the Newton line. Analogously point $N_1$ will lie on the polar $P_2Q_2$ of $E$ w.r. to $\kappa_1$.

Nr-2 is essentially equivalent to corollary 3.3 and derives from $N_1$ being the middle of $I_1J_1$, so that $E_1N_1$ passes also from the middle of the chord $C'D'$. This shows the stated property for $E_1E'$. Analogously is proved the property for $E_2E'$.

For nr-3 apply corollary 3.3 and take into account nr-2. By these the polar $\varepsilon$ of $E'$ must go through $E$, since $E'$ is on the polar of $E$. Since for all points $E'$ on $E_1N_1$ the
corresponding polars are parallel to the Newton line \( \nu \) it follows that \( \varepsilon \) is the parallel from \( E \) to \( \nu \). Analogously is seen that \( \varepsilon \) is also the polar of \( E' \) w.r. to \( \kappa_2 \).

Nr-4 follows from corollary 3.3 for \( R_2 \), which is on the polar \( \varepsilon \) of \( E' \), hence its polar, which is parallel to \( \nu \) must pass through \( E' \). The same corollary adapted for the parabola \( \kappa_2 \) implies the statement for \( R_1 \).

In nr-5, the statement for the middle \( M \) follows from nr-3, since by this \( R_1 = (P_1Q_1, \varepsilon) \) is harmonic conjugate to \( E' \) w.r. to \( \{P_1, Q_1\} \). Thus, \( E(P_1Q_1E'R_1) \) is a harmonic pencil and \( GH \) is parallel to its ray \( \varepsilon = ER_1 \), hence the other rays pass through the end points of \( GH \) and its middle \( M \). Since \( R_1 \) is on the polar of \( E \) and also of \( E' \), w.r. to \( \kappa_1 \), its polar is \( EE' \). Analogous is the proof for \( R_2 \). The last statement is an immediate consequence of the previous statements.

In nr-6, the first claim follows by observing that \( E(G, H, E', R_1) \) is a harmonic pencil and \( GH \) is parallel to its ray \( ER_1 \), hence \( M \) is the middle of \( GH \). Thus, the diagonals of the quadrangle \( EGE'H \) intersect at their middle. Analogous is the proof for the other claims. In fact, by lemma 3.4 the pencil \( E'(N_1, N_2, G, H) \) is harmonic and its ray \( E'H \) is parallel to \( AC \). Thus the other three rays intersect \( AC \) in two equal segments. Analogously is proved also that \( H \) is the middle of \( Q_1Q_2 \).

Nr-7 follows directly from nr-6, since \( \{P_1P_2, AC\} \) have \( G \) as a common middle etc.

For nr-8 start by defining \( P_H \) as the intersection \( P_H = (P_1N_1, \varepsilon) \) (See Figure 9). Since \( P_H \) is on the polars w.r. to \( \kappa_1 \) of \( G \) and \( E' \), its polar is \( GE' \). From lemma 3.4 follows that the pencil \( E'(GHN_1N_2) \) is harmonic, hence intersects on \( P_1N_1 \) a harmonic quadruple \( (P_1N_1G_1P_H^*) = -1 \). Since \( GE' \) is the polar of \( P_H \) the quadruple \( (P_1N_1G_1P_H) = -1 \) is also harmonic, hence \( P_H^* = P_H \). This completes the proof that \( P_H = (E'H, \varepsilon) \) and
its polar w.r. to $\kappa_1$ is $GE'$. By replacing in the previous argument points \{ $P_1, N_1, G_1$ \}, with the points \{ $P_2, N_2, G_2 = (P_2 N_2, GE')$ \}, we show analogously that $GE'$ is the polar of $P_H$ w.r. to $\kappa_2$. An analogous reasoning shows that $HE'$ is the polar of $P_G$.

\[\text{Figure 10. The circle } \lambda \text{ through the focal points}\]

Nr-9 is an immediate consequence of the previous nr. It reflects also the property of parabolas, according to which, the anticomplementary of a tangential triangle of a parabola is a self-polar triangle of the parabola ([2, p. 140], see also [1, p. 74]). □

**Corollary 3.6.** With the notation and conventions of this section, the circle $\lambda$ through the intersection $E$ of the diagonals and the middles $\{G, H\}$ of them, passes also through the focal points $\{F_1, F_2\}$ of the two parabolas $\{\kappa_1, \kappa_2\}$. Further the angle $\widehat{F_1 GF_2}$ equals one of the angles formed by the axes of the parabolas.

**Proof.** The first claim follows from the well known property of triangles circumscribing a parabola, since the triangle $EGH$ circumscribes both parabolas (See Figure 10). The second claim follows from the equally well known optical property of conics ([1, p. 6]). In fact, by theorem 3.5/nr-2 the lines $\{P_1 Q_1, P_2 Q_2\}$ are respectively parallel to the axes of $\{\kappa_2, \kappa_1\}$. Drawing $GF_2'$ parallel to $P_1 Q_1$ and $GF_1'$ parallel to $P_2 Q_2$ and applying the aforementioned property we obtain the relations

$$\widehat{F_2 GH} = \widehat{P_2 GF_2'} \quad \text{and} \quad \widehat{HGF_1} = \widehat{F_1' GP_1},$$

which imply immediately the claim. □

4. A PAIR OF EQUAL CIRCLES

Here we start by introducing the circle $\lambda'$ on diameter $E'K$, where $K$ is the intersection point of the directrices $\{\delta_1, \delta_2\}$ of the two parabolas $\{\kappa_1, \kappa_2\}$ (See Figure 11). Since, by theorem 3.5/nr-2, the lines $\{P_1 Q_1, P_2 Q_2\}$ are respectively parallel to the axes of the parabolas $\{\kappa_2, \kappa_1\}$, the angles at the intersection points $S_1 = (P_1 Q_1, \delta_2)$ and $S_2 = (P_2 Q_2, \delta_1)$ are right. Hence points $\{S_1, S_2\}$ are on the circle $\lambda'$ with diameter $E'K$.  

\[\text{Figure 11. The circle } \lambda' \text{ through the focal points}\]
Lemma 4.1. The pairs of points \{(F_1, S_2), (F_2, S_1)\} are symmetric w.r. to the Newton line \(\nu\) and the circles \{\(\lambda, \lambda'\)\} are congruent and intersect at points \{G, H\} of the Newton line.

Proof. The first claim follows from the characteristic property of a parabola. In fact, the contact point \(N_2\) of the tangent \(\nu\) to \(\kappa_2\) is equidistant from the points \{\(F_2, S_1\)\} and similarly, the contact point \(N_1\) of the tangent \(\nu\) to \(\kappa_1\) is equidistant from \{\(F_1, S_2\)\}, therefore the claimed symmetry, implying that \(\nu\) is the common medial line of the two parallel segments \{\(F_1S_2, F_2S_1\)\}.

The second claim follows from the first immediately, since \(F_1S_2F_2S_1\) is then an isosceles trapezium having equal diagonals \(S_1S_2 = F_1F_2\). By corollary 3.6, points \{\(K, E\)\} view respectively these equal segments under the same angle, hence the circles \(\lambda = (EF_1F_2)\) and \(\lambda' = (KS_1S_2)\) are equal.

That point \(H\) is on both circles, follows by observing the equal angles \(\widehat{F_1HF_2} = \widehat{S_1HS_2}\). Since the first characterizes the points of \(\lambda\) viewing \(F_1F_2\), the second angle will characterize also the points of the equal circle \(\lambda'\) viewing \(S_1S_2 = F_1F_2\). Thus \(H\) belongs also to \(\lambda'\). Analogously is seen that \(G\) belongs to the circle \(\lambda'\). \(\square\)

The circles \{\(\lambda, \lambda'\)\} are related to two triangles naturally defined in the configuration of the two parabolas. These are the triangles \{\(D_1D_2E', D_1D_2L\)\}, for the intersections of lines \{\(D_1 = (\delta_1, P_1Q_1), D_2 = (\delta_2, P_2Q_2)\)\} and \(L = (D_1F_1, D_2F_2)\) (See Figure 11). The following theorem lists some basic properties of these triangles.

Theorem 4.2. Under the notation and conventions adopted so far, the following are valid properties.

1. Lines \(\{EF_1, EF_2\}\) are polars respectively of \(D_1\) rel. \(\kappa_1\) and of \(D_2\) rel. \(\kappa_2\).
2. Lines \(\{D_1F_1, D_2F_2\}\) are respectively orthogonal to \(\{EF_1, EF_2\}\).
3. The quadrangle \(D_1D_2E'L\) is cyclic.
(4) Lines \( \{KE', EL\} \) are parallel and orthogonal to \( D_1D_2 \).

(5) Points \( \{K, E\} \) are orthocenters respectively of the triangles \( \{D_1D_2E', D_1D_2L\} \) and lines \( \{EF_1, EF_2\} \) pass respectively through points \( \{D_2, D_1\} \).

(6) The Newton line \( \nu \) passes through the middle \( M \) of \( D_1D_2 \) and the circles \( \{\lambda, \lambda'\} \) are orthogonal to the circle with diameter \( D_1D_2 \), consequently

\[
MG \cdot MH = MF_1^2 = MF_2^2 = D_1D_2^2/4.
\]

Proof. Nr-1, the claim for the line \( EF_1 \), follows from the definition of \( D_1 \) as intersection of \( P_1Q_1 \), which is the polar of \( E \) rel. \( \kappa_1 \), and of \( \delta_1 \), which is the polar of \( F_1 \). Hence \( EF_1 \) is the polar of \( D_1 \) rel. \( \kappa_1 \). Analogous is the proof of the claim for \( EF_2 \).

Nr-2 follows from the general property of the directrix, according to which the polar \( EF_1 \) w.r. to a point \( D_1 \) of the directrix is orthogonal to \( D_1F_1 \) ([11, p. 183]). Analogously is proved that \( EF_2 \) is orthogonal to \( D_2F_2 \).

Nr-3 is an immediate consequence of nr-2.

Nr-4 follows by observing that the orthogonal triangles \( \{KE'H, LGE\} \) are equal. In fact, by lemma 4.1, they have equal hypotenuses \( KE' = EL \), which are diameters of equal circles. Also, by theorem 3.5/nr-6, \( EGE'H \) is a parallelogram, hence the sides \( \{EG, HE'\} \) of these triangles are equal. The equality of these triangles implies the equality of the triangles \( \{KH, LHG\} \) and this implies the claimed parallelity of \( \{KH, EL\} \). The orthogonality of these two lines to \( D_1D_2 \) follows from the fact, noticed at the beginning of this section, that the directrices \( \{\delta_1, \delta_2\} \) are respectively orthogonal to \( \{D_2H, D_1E'\} \). Hence they are altitudes of the triangle \( D_1D_2E' \) and \( K \) is the orthocenter of the triangle. This implies that \( KE' \) is the third altitude of this triangle, consequently this and its parallel \( EL \) are both orthogonal to \( D_1D_2 \).

Nr-5, the proof for \( K \) was given in the course of proof of nr-4. The claim for \( E \) follows from the next lemma on triangles with the same circumcircle and same basis.

Nr-6 follows also from the next lemma.

\[\square\]

**Figure 12. Orthocenters \( \{E\} \) of circles on same circumcircle \( \kappa \) and base \( BC \)**

**Lemma 4.3.** The orthocenters \( \{H\} \) of triangles \( ABC \) with the same circumcircle \( \kappa \) and same base \( BC \) define congruent to each other circles \( \{\lambda\} \) with equal and parallel to each other diameters \( \{AH\} \). The circles \( \{\lambda\} \) are orthogonal to the circle \( \kappa' \) on diameter \( BC \) and the radical axis \( IJ \) of each pair of them passes through the middle \( M \) of \( BC \).
Proof. For the proof draw the two segments from $M$ to the intersections of the circles $\{\lambda\}$ with $\kappa'$ and show that they are tangent to the circles $\{\lambda\}$ and the angle between them is constant and independent of the position of $\lambda$. The details are left as an exercise. \(\square\)

5. Polars relative to the two parabolas

The polars of points relative to the two parabolas have some interesting properties, which we discuss in this section.

![Figure 13: Polars of points relative to the two parabolas](image)

**Lemma 5.1.** For the points $\{X, Y\}$ lying respectively on the lines $\{P_1Q_1, P_2Q_2\}$ the polars $\{\sigma_X, \tau_X\}$ of $X$ and the polars $\{\sigma_Y, \tau_Y\}$ relative to the parabolas $\{\kappa_1, \kappa_2\}$ have the following properties (See Figure 13).

1. All lines $\{\sigma_X, \sigma_Y\}$ pass through $E = (AC \cap BD)$.
2. If $Y = (P_2Q_2, \sigma_X)$, then $X = (P_1Q_1, \sigma_Y)$ and vice versa.
3. Lines $\{\tau_X, \tau_Y\}$ pass respectively through $\{Y, X\}$ and are parallel to the Newton line $\nu$.
4. The middle $M$ of the segment $XY$ is on the Newton line $\nu$.

**Proof.** Nr-1 follows from the pole-polar reciprocity. By this, since $X$ is on the polar of $E$ rel. $\kappa_1$, the polar $\sigma_X$ will pass through $E$. Analogous is the proof for $\sigma_Y$.

Nr-2 follows again from the same principle, since $Y$ being on the polar $\sigma_X$ of $X$ implies that the polar $\sigma_Y$ of $Y$ passes through $X$.

Nr-3 for line $\tau_X$ follows from fact that $P_1Q_1$ is parallel to the axis of the parabola $\kappa_2$ and intersects it at its contact point $N_2$ with the Newton line $\nu$. Hence this line is in the conjugate direction of $P_1Q_1$ relative to $\kappa_2$ and all polars of $X \in P_1Q_1$ are parallel to $\nu$. Analogous is the proof for $\tau_Y$.

Nr-4 follows from a basic property of parabolas, according to which the polar $\tau_X$ of $X$ is parallel and at double the distance of $X$ from the tangent $\nu$ at the intersection point.
\( N_2 \) of the parabola with the parallel \( P_1Q_1 \) to the axis of the parabola through \( X \) ([13, p. 58]).\]

\[\text{Figure 14. Homography } X \mapsto Y \text{ between two lines}\]

**Lemma 5.2.** With the notation and conventions of this section, the following are valid properties.

1. The map \( f : X \mapsto Y \) of line \( P_1Q_1 \) the line \( P_2Q_2 \), defined in the previous lemma, is a linear homographic relation between the two lines.
2. The lines \( XY \) envelope a parabola \( \kappa_3 \).
3. The parabola \( \kappa_3 \) is tangent to the 7 lines: \( \{E_1E_2, D_1D_2, AC, BD, P_1Q_1, P_2Q_2, \nu\} \).
4. On every tangent of \( \kappa_3 \) the previous 7 lines define respectively points \( \{V, U, Z, N, X, Y, M\} \), such that the ratios defined by any three of them is constant. In particular, \( E_2D_1/D_1P_1 = E_1D_2/D_2P_2 \).

**Proof.** Nr-1 follows from the linearity of the “polarity”, mapping a point \( X \) to its polar \( \sigma_X \) relative to a conic \( \kappa \). This is realized by using a system of homogeneous coordinates ([14, p. 174, I]), in which the conic is represented by a symmetric matrix \( A \) and the associated bilinear form \( x^T Ax = 0 \). Then, for a point \( X = x_0 \), the corresponding polar \( \sigma_X \) is represented by the equation \( x_0^T Ax = 0 \) ([14, p. 274, I], [12, p. 188]). And it is easily seen that restricting \( X = x_0 \) on a line, like \( \alpha = P_1Q_1 \) and taking the intersections \( Y \) of the polars \( \sigma_X : x_0^T Ax = 0 \) with a line \( \beta = P_2Q_2 \), defines a linear homography \( f : X \mapsto Y \), represented in line coordinates through a function \( f(x) = ux + v \), as stated.

Nr-2 is an immediate consequence of nr-1, the arguments and the references being the same with those used in section 3 for the establishment of the existence of the two parabolas \( \{\kappa_1, \kappa_2\} \).
Nr-3 follows by driving \( X \) to special places on the line \( P_1Q_1 \) and considering the intersection \( Y \) of its polar \( \sigma_X \) with \( P_2Q_2 \). In this respect it is useful, inspecting again figure 13, to notice the coincidences:

- If \( X = E_2 \), then \( Y = E_1 \), \( \sigma_X = EE_1 \), \( \sigma_y = EE_2 \), \( \tau_X \parallel \nu \) through \( E_1 \).
- If \( X = D_1 \), then \( Y = D_2 \), \( \sigma_X = ED_2 \), \( \sigma_y = ED_1 \), \( \tau_X \parallel \nu \) through \( D_2 \).
- If \( X = P_1 \), then \( Y = P_2 \), \( \sigma_X = \sigma_Y = P_1P_2 = AC \), \( \tau_X \parallel \nu \) through \( P_2 \).
- If \( X = Q_1 \), then \( Y = Q_2 \), \( \sigma_X = \sigma_Y = Q_1Q_2 = BD \), \( \tau_X \parallel \nu \) through \( Q_2 \).
- If \( X = N_2 \), then \( Y = N_1 \), \( \tau_X = \tau_Y = \nu \).

For example, when \( X = E_2 \), then, as noticed at the beginning of section 3, the polar \( \sigma_X \) being in that case the line \( E_1E \), we see that \( Y = E_1 \), so that \( E_1E \) is tangent to \( \kappa_3 \). Analogous are the proofs for the other lines of this list of coincidences.

Nr-5 is again a consequence of the fundamental property of parabolas for the segments intercepted on a tangent by three other tangents of it ([4, p. 52]).

6. ONE-PARAMETER-FAMILY OF QUADRANGLES

The structure of the three parabolas \( \{\kappa_i\} \), discussed in the previous section, allows the determination of a one-parameter-family of quadrangles, containing the quadrangle of reference and having these parabolas playing the same role for all its members. To see this, we continue with the notation of the previous section and use the various tangents \( \varepsilon_X \) of the parabola \( \kappa_3 \). These lines define through their intersections with the lines \( \{P_1Q_1, P_2Q_2\} \) the points \( \{X = E_2, Y = E_1\} \) (See Figure 15). By lemma 5.2, the parallel-
By the aforementioned lemma, the quadrangle $q_X = ABCD$ with vertices at the intersections of the four lines $\{E_1C_1, E_1C_2, E_2A_1, E_2A_2\}$ has the three parabolas $\{\kappa_i\}$ playing the roles analyzed in the previous sections. We summarize these results in the following main theorem.

**Theorem 6.1.** The diagonals $\{AC, BD\}$ of the quadrangle of reference define through their contact points with the parabolas $\{\kappa_1, \kappa_2\}$ respectively lines $\{P_1Q_1, P_2Q_2\}$ and the tangents $\{\varepsilon_X\}$ to the parabola $\kappa_3$ define on these lines the intersection points $\{E_1, E_2\}$ of opposite sides of a one-parameter-family of quadrangles $\{q_t\}$ with the following properties:

1. The family contains the quadrangle of reference $q$ as a special member.
2. All the quadrangles $q_t$ have their two pairs of opposite sides tangent respectively to the two parabolas $\{\kappa_1, \kappa_2\}$.
3. The quadrangles $\{q_t\}$ besides the common diagonal lines, share in addition the same Newton line and the same ratio of diagonals.

Notice that the construction of the tangent $\varepsilon_X$ of the parabola $\kappa_3$, from which depends the whole configuration of the family, can be done by starting with an arbitrary point $X = E_2$ on $P_1Q_1$. In fact, by lemma 5.2/nr-4, the equality of the ratios $E_2P_1/P_1Q_1 = E_1P_2/P_2Q_2$, allows the definition of the corresponding point $Y = E_1$ on $P_2Q_2$ and with this the tangent $\varepsilon_X = E_2E_1$ of $\kappa_3$.

![Figure 16. Family of quadrangles defined by lines $E_1E_2$ tangent to $\kappa_3$](image)

Figure 16 emphasizes the common elements to all members of the family. Among these elements are: the three parabolas, their common circumscribing triangle $EGH$, the diagonal lines and the ratio $r = AC/BD$, the Newton line $\nu$, the lines $\{P_1Q_1, P_2Q_2\}$ carrying the contact points of the parabolas with the sides of the triangle $EGH$ and the points $\{D_1, D_2\}$, which are intersections of the directrices of the parabolas with the lines $\{P_1Q_1, P_2Q_2\}$. By lemma 5.2/nr-3, the line $D_1D_2$ is also tangent to $\kappa_3$, thus defining a particular member of the family. Next theorem shows that this is the unique cyclic member of the family.
Theorem 6.2. The one-parameter-family contains precisely one cyclic quadrangle, defined by that tangent $\varepsilon_X$ of the parabola $\kappa_3$, which coincides with the line $D_1D_2$.

Proof. Most of the work for the proof has been done in section 4. There it was shown that the triangle $D_1D_2L$, where $L = (D_1F_1, D_2F_2)$, has $E$ as its orthocenter and the circle $\lambda = (EGH)$, carrying also the foci $\{F_1, F_2\}$ of the parabolas, intersects the circle $\kappa$, having $D_1D_2$ for diameter, orthogonally at these two points (See Figure 17). In [8] was shown that this is a necessary and sufficient condition for $ABCD$ to be cyclic. Thus, for $\varepsilon_X = D_1D_2$ the corresponding quadrangle is indeed cyclic.

To show that this is the unique genuine cyclic quadrangle contained in the family, we see first that there is a second position of the tangent $\varepsilon_X$, which has a corresponding circle $\kappa$ on diameter $E_1E_2$ also orthogonal to $\lambda$. This happens when $\varepsilon_X$ coincides with the Newton line $\nu$, in which case $E_1E_2 = N_1N_2$. But, as can easily deduced from lemma 3.4 and theorem 4.2, the circle with diameter $N_1N_2$ is orthogonal to $\lambda$, and the corresponding quadrangle degenerates to a line segment. Thus, taking into account the last cited reference, in order to show that there is no other genuine cyclic quadrangle of the family, it suffices to show that from all these circles with diameter $E_1E_2$ on corresponding tangents $\varepsilon_X$ of $\kappa_3$, only the two aforementioned circles with corresponding diameters $\{D_1D_2, N_1N_2\}$ are orthogonal to the circle $\lambda$.

This would be easy to show if the circles $\kappa$, whose centers are on $\nu$, were members of a pencil ([9, p. 106]). But they are not, and, in order to prove this, we must resort to a short calculation. This can be done as follows. Referring to figure 16, we recall that the diameters $E_1E_2$ of all these circles, are defined by considering an arbitrary point $X = E_2$ on $P_1Q_1$, which is a fixed tangent of $\kappa_3$ and taking $Y = E_1$ on the other fixed tangent $P_2Q_2$, such that $E_2P_1/P_1Q_1 = E_1P_2/P_2Q_2$. Considering now the oblique coordinate system of lines $\{\nu, Q_1P_1\}$ intersecting at $N_2$, we see easily that the coordinates $\{M(x), E_2(y)\}$ of the middle $M$ of $E_1E_2$ and of $E_2$ satisfy a linear relation $y = ax + b$, with constants $\{a, b\}$. Let us denote now by $\kappa_x$ and $M_x$ the variable circle with diameter $E_1E_2$ and its center $M$. Let also $f$ denote the inversion relative to the circle $\lambda(O)$. In
order to have orthogonality of $\kappa_x$ on $\lambda$, it is necessary and sufficient that the centers $\{M_x, N_x\}$ of the circles $\{\kappa_x, \nu_x = f(\kappa_x)\}$ coincide.

Figure 18 illustrates the proof idea. The centers $N_x$ of the circles $\nu_x = f(\kappa_x)$ describe a conic. Having found already two intersection points $\{M', M''\}$ of the geometric locus with $\nu$, which are the centers of the two circles with diameters $\{D_1D_2, N_1N_2\}$, if we prove that the centers $\{N_x\}$ describe a conic, then there can be no other intersections with the line $\nu$ and the theorem is proved.

That points $\{N_x\}$ describe a conic, can be proved by using the easily deducible relation between the centers $M_x$ and $N_x$ of the circles:

\[(2) \quad N_x = O + r^2 \frac{OM_x}{|OM_x|^2 - r_x^2},\]

where $\{r, r_x\}$ are respectively the radii of $\{\lambda, \kappa_x\}$. Introducing unit vectors $\{u, v\}$ along the lines $\{\nu, E'P_1\}$, we can represent the points in the form

\[M_x = x \cdot u, \quad E_2 = (ax + b) \cdot v, \quad O = s \cdot u + t \cdot v.\]

Introducing these into equation 2, we find, after simplification, that

\[(3) \quad \frac{ON_x}{OM_x} = \frac{r^2}{Ax^2 + Bx + C},\]

for some appropriate constants $\{A, B, C\}$. The proof of the theorem is completed by the following lemma, whose proof is left as an exercise. □

**Lemma 6.3.** Given a line $\varepsilon$, a point $O \notin \varepsilon$ and a variable point $M_x \in \varepsilon$, the point $N_x$ lying on the line $OM_x$ and satisfying

\[\frac{ON_x}{OM_x} = \frac{1}{Ax^2 + Bx + C},\]

for some constant numbers $\{A, B, C\}$ and a system of line coordinates $\{x\}$ for $\varepsilon$, describes a conic, passing through $O$ and having there a tangent parallel to $\varepsilon$.  

Figure 19. Conic generation through a quadratic function

REFERENCES


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