Abstract. We generalize two sangaku problems involving an arbelos proposed by Izumiya and Naitō, and show the existence of six non-Archimedean congruent circles.

1. Introduction

In this article we generalize two sangaku problems involving an arbelos proposed by Izumiya and Naitō. Let α, β and γ be the three semicircles with diameters AO, BO and AB, respectively for a point O on the segment AB constructed on the same side of AB. The area surrounded by the three semicircles is called arbelos (see Figure 1). The radical axis of α and β is called the axis. Let a and b be the radii of α and β, respectively, and let δ_α (resp. δ_β) be the incircle of the curvilinear triangle made by α (resp. β), γ and the axis. The two circles δ_α and δ_β have common radius r_A = ab/(a + b) and are called the twin circles of Archimedes.

Izumiya’s problems appeared in a sangaku in Saitama hung in 1866, which is as follows [6] (see Figure 2).

Problem 1. If α and β are congruent and the tangent of α from B meets γ in a point C, show that the inradius of the curvilinear triangle formed by α, γ and the perpendicular from C to AB equals a/9.
Naitō’s problem appeared in a sangaku in Fukushima hung in 1983 (the sangaku seems to be made in modern day times), which is as follows [1] (see Figure 3).

**Problem 2.** If $\alpha$ and $\beta$ are congruent, show that the radius of the circle touching the remaining external common tangent of $\alpha$ and $\delta_\alpha$ and the arc of $\gamma$ cut by the tangent at the midpoint equals $a/9$.

2. **Generalization**

We now consider the case in which the semicircles $\alpha$ and $\beta$ are not always congruent. We use the next proposition (see Figure 4).

**Proposition 2.1.** For a point $P$ on the segment $AB$, let $h$ be the perpendicular to $AB$ at $P$. If $\delta_1$ is the circle touching $h$ at $P$ from the side opposite to $B$ and the tangent of $\beta$ from $A$ and $\delta_2$ is the circle touching $\alpha$ externally $\gamma$ internally and $h$ from the same side as $\delta_1$, then $\delta_1$ and $\delta_2$ are congruent.

**Proof.** The radius of $\delta_2$ is proportional to the distance between its center and the radical axis of $\alpha$ and $\gamma$ [2, p. 108], while $\delta_2$ coincides with $\beta$ if $P = B$. Also the radius of $\delta_1$ is proportional to the distance between its center and the point $A$, and $\delta_1$ coincides with $\beta$ if $P = B$.  

□
Theorem 2.2. Let $C$ be the point of intersection of $\gamma$ and the tangent of $\alpha$ from $B$ and let $D$ be the foot of perpendicular from $C$ to $AB$. The incircle of the curvilinear triangle made by $\alpha$, $\gamma$ and $CD$ is denoted by $\varepsilon_1$. Let $u$ be the remaining external common tangent of $\alpha$ and $\delta_\alpha$. The circle touching $u$ and the arc of $\gamma$ cut by $u$ at the midpoint is denoted by $\varepsilon_2$. The incircle of the curvilinear triangle made by $\gamma$, $\delta_\beta$ and the axis is denoted by $\varepsilon_3$. The circle touching the tangent of $\beta$ from $A$ and $CD$ at $D$ from the side opposite to $B$ is denoted by $\varepsilon_4$. The smallest circle passing through the point of intersection of $BC$ and $u$ and touching the line $CD$ is denoted by $\varepsilon_6$. Then the following statements hold.

(i) The six circles $\varepsilon_1$, $\varepsilon_2$, $\ldots$, $\varepsilon_6$ are congruent and have common radius

$$\frac{a^2b}{(a+2b)^2}.$$ 

(ii) The circle $\varepsilon_1$ touches the line $t$, and the circle $\varepsilon_2$ touches $\gamma$ at $C$.

Proof. We assume that $r_i$ is the radius of $\varepsilon_i$, $d = a + 2b$, $E$ is the point of intersection of $BC$ and $\beta$, $F$ is the foot of perpendicular from $E$ to the axis, $G$ is the point of tangency of $\alpha$ and $BC$, $H$ is the center of $\alpha$, and $BC$ meets the axis and $u$ in points $J$ and $K$, respectively (see Figure 6).

Since the three segments $CA$, $GH$ and $EO$ are parallel and $H$ is the midpoint of $AO$, $G$ is the midpoint of $CE$. While the line $BC$ is the internal common tangent of $\alpha$ and $\delta_\alpha$ [3, p. 212]. Therefore $G$ is also the midpoint of $JK$. Hence $|EJ| = |CK|$, i.e., the circles $\varepsilon_5$ and $\varepsilon_6$ are congruent. Since the triangles $BGH$, $BEO$ and $OFE$ are similar, $a/d = |OE|/|2b| = |EF|/|OE|$. Therefore $|OE| = 2ab/d$ and $|EF| = 2a^2b/d^2$. Hence $r_5 = a^2b/d^2 = r_6$, and $|OF| = 4ab\sqrt{(a+b)b/d^2}$ from the right triangle $OFE$. 
The last equation implies $|EF| = \frac{a|OF|}{2\sqrt{(a+b)b}}$. Let $x = |BD|$. Then $|CD| = \frac{ax}{2\sqrt{(a+b)b}}$ from the similar triangles $OFE$ and $BDC$. Therefore we have

$$x(2(a + b) - x) = |CD|^2 = \frac{a^2x^2}{4(a+b)b}.$$ Solving the equation for $x$, we get $x = 8b(a+b)^2/d^2$. Therefore $|AD| = 2(a+b) - x = 2a^2(a+b)/d^2$.

Therefore $r_4 = b|AD|/|AB| = a^2b/d^2 = r_1$ by Proposition 2.1. Meanwhile $\varepsilon_3$ and the incircle of the curvilinear triangle made by $\alpha, \gamma$ and $t$ have radius $a^2b/d^2$ [5, Theorem 9]. Therefore the last circle coincides with $\varepsilon_1$, i.e., $\varepsilon_1$ touches $t$. While we have also shown that $\varepsilon_1$ and $\varepsilon_2$ are congruent in [4]. This proves (i) and the first half part of (ii).

Let $\zeta$ be the circle with center $C$ passing through $G$. We invert the figure in $\zeta$. Then the circles $\alpha$ and $\delta_\alpha$ are orthogonal to $\zeta$, i.e., they are fixed by the inversion. The line $u$, which intersects $\zeta$, is inverted to a circle intersecting $\zeta$ touching $\alpha$ and $\delta_\alpha$ passing through $C$. Therefore $\gamma$ is the inverse of $u$. This implies that the points of intersection of $\gamma$ and $u$ lie on $\zeta$. Hence $C$ is the midpoint of the arc of $\gamma$ cut by $u$. Therefore $\varepsilon_2$ touches $\gamma$ at $C$. This proves the second half part of (ii). □

Circles of radius $r_\Lambda$ are called Archimedean circles [3]. Therefore we now have six non-Archimedean congruent circles $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_6$. Exchanging the roles of $\alpha$ and $\beta$, we get another six non-Archimedean congruent circles of radius $ab^2/(2a+b)^2$, which are denoted in Figure 5.

3. The circle associated with a point on $\gamma$

For a circle $\delta$ touching $\alpha$ externally and $\gamma$ internally, if $P$ is the point of intersection of $\gamma$ and the internal common tangent of $\delta$ and $\alpha$ closer to $B$, we say that $\delta$ is associated with $P$. As mentioned in the proof of Theorem 2.2, the circle $\delta_\alpha$ is associated with the point $B$ (see Figure 6). We can also consider that the point circle $A$ is associated with the point $A$ itself, because the perpendicular to $AB$ at $A$ can be considered as the internal common tangent of the point circle $A$ and $\alpha$. Let $I$ be the point of intersection of $\gamma$ and the axis. The next theorem gives the circle associated with the point $I$.
Theorem 3.1. The internal common tangent of \( \alpha \) and \( \varepsilon_1 \) passes through \( I \).

Proof. Let \( \rho \) be the circle with center \( I \) passing through \( O \). We invert the figure in \( \rho \) (see Figure 7). Then \( \alpha \) and \( \beta \) are fixed. While \( t \), which intersects \( \rho \), is inverted into the circle with center \( I \) touching \( \alpha \) and \( \beta \) intersecting \( \rho \). Therefore \( \gamma \) is the inverse of \( t \). Hence the figure consisting of \( \alpha \), \( \gamma \) and \( t \) is fixed by the inversion. This implies that \( \varepsilon_1 \) is also fixed. Since \( \alpha \) and \( \varepsilon_1 \) are orthogonal to \( \rho \), their point of tangency lies on \( \rho \), and their common internal tangent passes through \( I \).  

\[ \Box \]

The proof also shows that the points of intersection of \( \gamma \) and \( t \) lies on \( \rho \). Therefore \( I \) is the midpoint of the arc of \( \gamma \) cut by \( t \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{Figure 7.}
\end{figure}

References


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