ON BROCARD’S POINTS IN POLYGONS

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Abstract. In this note we present a synthetic proof of the key lemma, defines in the problem of A. A. Zaslavsky.

For any given convex quadrilateral $ABCD$ there exists a unique point $P$ such that $\angle PAB = \angle PBC = \angle PCD$. Let us call this point the Brocard point $(Br(ABCD))$, and respective angle — Brocard angle $(\phi(ABCD))$ of the broken line $ABCD$. You can read the proof of this fact in the beginning of the article by Dimitar Belev about the Brocard points in a convex quadrilateral [1].

![Fig. 1.](image)

In the first volume of JCGeometry [2] A. A. Zaslavsky defines the open problem mentioning that $\phi(ABCD) = \phi(DCBA)$ (namely there are such points $P$ and $Q$, that $\angle PAB = \angle PBC = \angle PCD = \angle QBA = \angle QCB = \angle QDC = \phi$, moreover $OP = OQ$ and $\angle POQ = 2\phi$ iff $ABCD$ is cyclic, where $O$ is the circumcenter of $ABCD$.

Synthetic proof of these conditions is provided below.

Proof. 1) We have to prove that if $\phi(ABCD) = \phi(DCBA)$, then $ABCD$ is cyclic. Let $P$ and $Q$ be $Br(ABCD)$ and $Br(DCBA)$ respectively. Let us denote angles $\phi(ABCD)$ and $\phi(DCBA)$ by $\phi$. We also denote $E = AP \cap BQ$, $G = BP \cap CQ$, and $F = CP \cap DQ$. Using $\angle PAB = \angle PBC = \angle PCD$ and $\angle QBA = \angle QCB = \angle QDC$ we obtain $\angle QEP = \angle AEB = \pi - 2\phi = \angle BGC = \angle GCP$. From this it follows that quadrilateral $QEGP$ is cyclic. Denote its circumcircle by $\omega$. Similarly we can prove that $F$ also belongs to $\omega$.

It remains to prove that the intersection of perpendicular bisectors of sides $AB$, $BC$, $CD$ lies on $\omega$. Let us consider the perpendicular bisector of $AB$. It intersects $\omega$ in point $K$, besides point $E$ lies on it because $\angle EAB = \phi = \angle EBA$. 


Then we have $\angle QEK = \frac{\pi}{2} - \phi = \angle PKE$. Therefore point $K$ is the middle of arc $PQ$ and all perpendicular bisectors pass through the middle of arc $PQ$ of $\omega$.

2) Given that quadrilateral $ABCD$ is cyclic, denote point $Br(ABCD)$ by point $P$ and angle $\phi(ABCD)$ by $\phi$. There exists point $Q$ that $\angle QBA = \angle QCB = \phi$ and point $D'$ on the line $CD$ that $\angle QD'C = \phi$. Then from the point 1) we get that quadrilateral $ABCD'$ is inscribed. This implies that points $D$ and $D'$ are the same and $\angle QDC = \phi$. □

**Remark.** From this it also follows that point $K$ is circumcenter of $ABCD$ and $KP = KQ$, $\angle PKQ = 2\phi$.

It turns out that the above statement may be also used with Brocard polygons.

Recall that the polygon $A_1A_2 \ldots A_n$ is called Brocard ones if it is cyclic and there exists a unique point $P$ such that $\angle PA_1A_2 = \angle PA_2A_3 = \ldots = \angle PA_nA_1 = \phi$. 
Let us prove that a polygon $A_1A_2\ldots A_n$ is the Brocard one if and only if there exists unique point $P$ such that $\angle PA_1A_2 = \angle PA_2A_3 = \ldots = \angle PA_nA_1 = \phi$ and a unique point $Q$ such that $\angle QA_2A_1 = \angle QA_3A_2 = \ldots = \angle PA_1A_n = \phi$.

Notice that we give generalization not for every inscribed polygon but only for Brocard ones.

**Proof.** 1) Assume that the polygon $A_1A_2\ldots A_n$ is Brocard. Since quadrilateral $A_1A_2A_3A_4$ is cyclic, it follows that there exists unique point $Q_1$ such that $\angle Q_1A_2A_1 = \angle Q_1A_3A_2 = \angle Q_1A_4A_3 = \phi$ and $OP = OQ_1, \angle POQ_1 = 2\phi$. Since quadrilateral $A_2A_3A_4A_5$ is cyclic, it follows that there exists point $Q_2$ such that $\angle Q_2A_3A_2 = \angle Q_2A_4A_3 = \angle Q_2A_5A_4 = \phi$ and $OP = OQ_2, \angle POQ_2 = 2\phi$. Obviously points $Q_1$ and $Q_2$ are the same. Therefore we obtain point $Q$ such that $\angle QA_2A_1 = \angle QA_3A_2 = \ldots = \angle QA_1A_n = \phi$.

2) Given polygon $A_1A_2\ldots A_n$ and points $P$ and $Q$ such that $\angle PA_1A_2 = \angle PA_2A_3 = \ldots = \angle PA_nA_1 = \phi = \angle QA_2A_1 = \angle QA_3A_2 = \ldots = \angle QA_1A_n$. Clearly that quadrilateral $A_iA_{i+1}A_{i+2}A_{i+3}$ is cyclic for all $i$’s. It follows that polygon $A_1A_2\ldots A_n$ is cyclic. \hfill \□

**References**


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