ON CIRCLES TOUCHING THE INCIRCLE

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Abstract. For a given triangle, we deal with the circles tangent to the incircle and passing through two its vertices. We present some known and recent properties of the points of tangency and some related objects. Further we outline some generalizations for polygons and polytopes.

1. Case of triangle

Throughout this section, we use the following notation\(^1\).

Let \(ABC\) be a triangle. The circles \(\gamma\) and \(\Gamma \) are its incircle and circumcircle with centers \(I\) and \(O\) and radii \(r\) and \(R\), respectively. The sides \(BC\), \(CA\), and \(AB\) touch \(\gamma\) at points \(A_1\), \(B_1\), and \(C_1\), respectively.

Now construct the circle \(\omega_A\) passing through \(B\) and \(C\) and tangent to \(\gamma\) at some point \(X_A\). Define the circles \(\omega_B\), \(\omega_C\) and the points of tangency \(X_B\), \(X_C\) in a similar way. This paper is devoted to the investigation of the objects related to the constructed circles.

Let \(M_A\) be the second meeting point of \(\omega_A\) and \(X_AA_1\); define the points \(M_B\) and \(M_C\) analogously.

We start with the following description of the points \(M_A\), \(M_B\), and \(M_C\).

**Theorem 1.** Line \(OM_A\) is the perpendicular bisector of \(BC\). Further, point \(M_A\) is the radical center of \(\gamma\), \(B\), and \(C\) (here we regard \(B\) and \(C\) as degenerate circles).

*Proof.* Consider the homothety with center \(X_A\) taking \(\gamma\) to \(\omega_A\). This homothety takes \(BC\) to the line \(t_A\) touching \(\omega_A\) at \(M_A\), hence \(t_A \parallel BC\). Thus \(M_A\) is the midpoint of arc \(BC\), and \(OM_A\) is the perpendicular bisector of \(BC\).

Consequently, there exists an inversion \(\iota\) with center \(M_A\) which takes \(BC\) to \(\omega_A\). We have \(\iota(A_1) = X_A\), \(\iota(B) = B\), \(\iota(C) = C\), hence \(M_AA_1 \cdot M_AX_A = M_AB^2 = M_AC^2\). This means that \(M_A\) has equal powers with respect to the circles \(\gamma\), \(B\), and \(C\). \(\square\)

\(^1\)All the results from this section allow various generalizations which are shown in the next section. So, throughout this section we put into footnotes the statements and approached that work only in this particular case, and fail up to further generalizations.
Remark 1. One could finish the proof without using inversion in the following way. In triangles $M_AB_1A$ and $M_AX_AB$ we have $\angle A_1BM_A = \frac{1}{2} \angle MA = \frac{1}{2} \angle MAB = \angle BXM_A$. Hence $\triangle M_AB_1A \sim \triangle M_AX_AB$, and $M_AA_1 \cdot M_AX_A = M_AB^2$.

Let $\ell_A$ be the radical axis of circles $\gamma$ and $A$; hence $\ell_A$ passes through the midpoints of $AB_1$ and $AC_1$. Define the lines $\ell_B$ and $\ell_C$ in a similar way. From Theorem 1 and analogous statements for $M_B$ and $M_C$, we obtain the following

Corollary 2. $\ell_A = M_BM_C$, $\ell_B = M_CM_A$, $\ell_C = M_AM_B$.

Theorem 3. The circumcenter of triangle $M_AM_BM_C$ is $O$. Next, triangles $M_AM_BM_C$ and $A_1B_1C_1$ are homothetical with some center $K$.

Proof. By Corollary 2, we have $M_BM_C \perp AI$. Next, by Theorem 1 we get $M_BO \perp AC$ and $M_CO \perp AB$. Consequently, $\angle OMB_C = \angle CAI = \angle IAB = \angle M_BMC_O$; therefore, $M_BO = M_CO$. Analogously, we get $M_AO = M_BO$, and the points $M_A$, $M_B$, and $M_C$ lie on some circle $\Omega$ with center $O$.

Consider now the homothety $h$ with positive coefficient which takes $\gamma$ to $\Omega$; thus $h(I) = O$. Notice that the vectors $\overrightarrow{IA_1}$ and $\overrightarrow{OM_A}$ are codirectional, so $h(A_1) = M_A$; analogously, $h(B_1) = M_B$ and $h(C_1) = M_C$, as desired.$^2$

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$^2$Let us present an alternative proof. For the second claim of Theorem, it suffices to show that the respective sides of triangles $M_AM_BM_C$ and $A_1B_1C_1$ are parallel. From Corollary 2 we get $\ell_B = M_AM_C$. Since $\ell_B \perp BI$, we have $\ell_B \parallel A_1C_1$. Similarly, $\ell_A \parallel B_1C_1$ and $\ell_C \parallel A_1B_1$. 

\[ \text{Fig. 1.} \]
Till the end of this section, we fix the notation of $\Omega$ as the circumcircle of $MA MB MC$, $h$ as the homothety taking $A_1B_1C_1$ to $MA MB MC$, and $K$ as its center. Denote by $R(\Omega)$ the radius of $\Omega$.

From the homothety $h$, we immediately obtain the following.

**Corollary 4.** Lines $A_1X_A = X_AMA$, $B_1X_B = X_BMB$, and $C_1X_C = X_CMC$ are concurrent at $K$.

**Corollary 5.** Points $K$, $I$, and $O$ are collinear, and $\frac{KO}{KI} = \frac{R(\Omega)}{r}$.

Now we describe point $K$ in some other terms.

**Theorem 6.** $K$ is the radical center of $\omega_A$, $\omega_B$, and $\omega_C$.

For the first part, consider the homothety $h$ taking $\triangle A_1B_1C_1$ to $\triangle MA MB MC$. Let $\Omega$ be the circumcircle of $MA MB MC$; then $h(\gamma) = \Omega$. Hence $h(I)$ is the center of $\Omega$. Next, $h$ takes $IA_1$ to the line passing through $MA$ and perpendicular to $BC$, which is the perpendicular bisector of $BC$. Similarly, $h(IB_1)$ is the perpendicular bisector of $AC$. Since the perpendicular bisectors of $BC$ and $AC$ pass through $O$, we get $h(I) = O$. 
Proof. From the homothety \( h \) we get \( \frac{KA_1}{KC_1} = \frac{KM_A}{KM_C} \). Hence the relation \( KX_A \cdot KA_1 = KX_C \cdot KC_1 \) implies \( KX_A \cdot KM_A = KX_C \cdot KM_C \). Thus we obtain the equality of powers of \( K \) with respect to \( \omega_A \) and \( \omega_C \). The same is true for \( \omega_A \) and \( \omega_B \). \( \square \)

Remark 2. In fact, this theorem is an instance of a more general fact. Consider two fixed circles \( \gamma \) and \( \Omega \), and let \( \omega \) be a variable circle tangent to \( \gamma \) at \( X \) and to \( \Omega \) at \( M \) (with a fixed type of tangencies — internal or external). Then, by Monge’s theorem, all the lines \( KM \) pass through a fixed point \( K \) which is a center of homothety taking \( \gamma \) to \( \Omega \). Moreover, \( K \) is the radical center of all such circles \( \omega \): if \( Y \) is the second common point of \( XM \) and \( \gamma \), then \( KX \cdot KM \) is proportional to \( KX \cdot KY \) which is fixed.

Remark 3. Theorem 6 combined with the first statement of Corollary 5 forms the statement of a problem on the Romanian Masters of Mathematics 2012 olympiad proposed by F. Ivlev [6, Problem 3].

Next, we introduce some more objects related to our construction. Let \( m_A \) be the line passing through \( A \) and perpendicular to \( IA \). Define lines \( m_B \) and \( m_C \)
similarly. Finally, let us define the points\(^3\) \(I_A = m_B \cap m_C, I_B = m_C \cap m_A,\) and \(I_C = m_A \cap m_B.\)

**Theorem 7.** \(M_A\) is the midpoint of the segment \(A_1I_A.\)

**Proof.** By its definition, line \(\ell_B\) is the midline between parallel lines \(A_1C_1\) and \(m_B;\) hence \(m_B\) intersects \(A_1M_A\) at point \(I'_A\) such that \(M_AI'_A = M_AA_1.\) Similarly, \(m_C\) also intersects \(A_1M_A\) at the same point, hence \(I'_A = I_A,\) and \(M_A\) is the midpoint of \(A_1I_A.\) \hfill \(\square\)

Let \(S_A\) be the circumcenter of triangle \(BIC.\) Define \(S_B\) and \(S_C\) similarly\(^4\). Denote by \(\Gamma_S\) the circumcircle of triangle \(S_AS_BS_C;\) let \(R_S\) be the radius of this circle\(^5\).

**Theorem 8.** The circle \(\Gamma_S\) has \(O\) as its center, and the equality \(R_S = R(\Omega) - r/2\) holds\(^6\). Moreover, triangles \(S_AS_BS_C\) and \(M_AM_BM_C\) are homothetical with center \(O.\)

\(^3\)Notice that \(m_A, m_B,\) and \(m_C\) are external bisectors of \(\angle A, \angle B,\) and \(\angle C,\) respectively, while \(I_A, I_B,\) and \(I_C\) are the excenters of \(\triangle ABC.\)

\(^4\)\(S_A, S_B,\) and \(S_C\) are the midpoints of arcs \(BC, CA,\) and \(AB\) of circle \(\Gamma,\) respectively.

\(^5\)In our case of triangle, \(\Gamma_S = \Gamma.\)

\(^6\)Thus \(R(\Omega) = R + r/2.\)
midpoint of $II_A$. Using Theorem 7, we conclude that $S_AM_A$ is the midline in triangle $I_AI_A1$, hence $\overrightarrow{S_AM_A} = \overrightarrow{IA_1}/2$. Since the vectors $\overrightarrow{OS_A}$ and $\overrightarrow{IA_1}$ are codirectional, we get $\overrightarrow{OS_A} = |\overrightarrow{OS_A}| = |\overrightarrow{OM_A}| - |\overrightarrow{IA_1}|/2 = R(\Omega) - r/2$. Similarly, we obtain that the segments $OS_B$ and $OS_C$ have the same length, so $O$ is the center of $\Gamma_S$. Finally, the homothety with center $O$ which takes $\Omega$ to $\Gamma_S$ takes $MA_MB_MC$ to $SA_SB_SC$.

Remarked 4. Using the relation between ratios of similarity of triangles $A_1B_1C_1$, $I_AI_IB_C$, $MA_MB_MC$, and $SA_SB_SC$, one may obtain the relation between radii in alternative way: we have $R(\Omega) = R(MA_MB_MC) = (R(I_AI_IB_C) + R(A_1B_1C_1))/2$ and $R(I_AI_IB_C) = 2R(SA_SB_SC) = 2R_S$; hence $R(\Omega) = R_S + r/2$.

Remark 5. The radius $R_S$ could be calculated in terms of $R$, $r$, and $OI$: $R_S = \frac{R^2 - OI^2}{2r}$ (see, for example, the Problem from Sharygin olympiad, 2005 [7, Problem 20]). Combining with Corollary 5 we get $\frac{KO}{K_T} = \frac{R^2 + r^2 - OI^2}{2r^2}$ thus obtaining the ratio of our homothety.  

At the end of this section, let us make some remarks on the results obtained above.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{Fig. 5.}
\end{figure}

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7By Euler formula, $OI^2 = R^2 - 2Rr$, hence $\frac{KO}{K_T} = \frac{R}{r} + \frac{1}{2}$
Remark 6. Here, we present an alternative approach to considered construction (without using points $M_A$, $M_B$, and $M_C$) related to a result by S. Ilyasov and A. Akopyan [5, Problem M2244]. We present here an equivalent reformulation of this statement; for the completeness, we also provide its proof.

**Proposition 1.** Let $\Gamma$ and $\gamma$ be two fixed circles, and let $A$, $B$ be variable points on $\Gamma$. Suppose two circles $\omega_1$ and $\omega_2$ (each may degenerate to a line) passing through $A$ and $B$ are tangent to $\gamma$ at $X$ and $Y$, respectively. Then the line $XY$ passes through a fixed point.

**Proof.** Let $\ell$ be the radical axis of $\gamma$ and $\Gamma$, and let $K$ be its pole with respect to $\gamma$. We claim that $XY$ passes through $K$.

Denote by $F$ the point of intersection of $AB$ with the common tangent to $\gamma$ and $\omega_1$ at $X$. Then $F$ is the radical center of $\gamma$, $\Gamma$, and $\omega_1$, hence it belongs to $\ell$. Since the powers of $F$ with respect to $\omega_1$ and $\omega_2$ are equal (in fact, they are equal to $FA \cdot FB$), this point lies also on the common tangent to $\gamma$ and $\omega_2$ at $Y$. Hence $XY$ is the polar line of $F$ with respect to $\gamma$, so it passes through $K$. □

In our case, one may apply this fact to the pairs of points $(A,B)$, $(B,C)$, and $(C,A)$ obtaining that the lines $X_A A_1$, $X_B B_1$, and $X_C C_1$ are concurrent at $K$. This proves Corollary 4 with one more description of point $K$. Next, since $\ell \perp OI$ and $IK \perp \ell$, we have $K \in OI$, thus proving Corollary 5. Note that from this new description of $K$ one could easily derive that $\frac{KO}{KI} = \frac{r^2 + r^2 - OI^2}{2r^2}$.

Further, let the tangents to $\gamma$ at $X_B$ and $X_C$ intersect at point $Z$; this point is the radical center of $\omega_B$, $\omega_C$, and $\gamma$. Hence $AZ$ is the radical axis of $\omega_B$ and $\omega_C$. To finish an alternative proof of Theorem 6, it suffices to show that $K \in AZ$. For that, notice that the point $Y$ of intersection of lines $B_1 C_1$ and $X_B X_C$ lies on the polar line $\ell$ of $K$ with respect to $\gamma$. Then the polar line of $Y$ contains the poles of lines $B_1 C_1$, $\ell$, and $X_B X_C$, which are $A$, $K$, and $Z$, respectively.

One may also notice that this approach allows to generalize some of the facts to the case of curved triangle $ABC$ (when its sides are the circular arcs). Namely, in this case the lines $A_1 X_A$, $B_1 X_B$, and $C_1 X_C$ are also concurrent at some point collinear with $O$ and $I$.

**Remark 7.** Let us mention two facts related to the considered construction.

a) The line containing points $A_1$, $I_A$, $M_A$, and $X_A$ passes also through the midpoint of the altitude from $A$ (see a problem on the Moscow mathematical olympiad 2001 [4, Problem 10.3], and also a problem proposed by Bulgaria for the IMO in 2002 [3, Problem 2002-G7, p. 319]). To prove this one can use the homothety with center $A$ taking the incircle to the excircle.

b) Lines $AX_A$, $BX_B$, and $CX_C$ are concurrent (see a problem by A. Badzyan in [1, Problem M2268]; this fact was independently noticed by D. Shvetsov).

**Remark 8.** All the results from this section could be reformulated if the incircle is replaced by one of the excircles.
2. A GENERAL CASE: POLYGONS AND POLYTOPES

All Theorems and Corollaries as well as remarks from the previous section admit generalizations to the case when the base figure is a polygon which is simultaneously circumscribed and inscribed, and also to the case when the base figure is a polytope in space (or in \(n\)-dimensional space) which is simultaneously circumscribed and inscribed.

Let us start from the general set up. Further we use the terminology for the case of 3-dimensional space though everything is appropriate for \(n\)-dimensional case for all \(n \geq 2\). For \(n = 2\) one could replace faces by sides, planes by lines, and spheres by circles. For \(n > 3\) one could replace faces by \((n-1)\)-dimensional hyperfaces, planes by hyperplanes, spheres by \((n-1)\)-dimensional spheres, and circles by \((n-2)\)-dimensional spheres.

Let \(\mathcal{P}\) be a convex polytope with vertices \(P_1, \ldots, P_n\); let \(F_1, \ldots, F_k\) be the planes determined by the faces of \(\mathcal{P}\). Suppose that \(\mathcal{P}\) has both an inscribed sphere \(\gamma\) and a circumscribed sphere \(\Gamma\). Let \(I, O\) and \(r, R\) be the centers and the radii of these spheres, respectively.

For every \(i \in \{1, 2, \ldots, k\}\), define \(\Gamma_i = F_i \cap \Gamma\) (thus, \(\Gamma_i\) is the circumcircle of the face lying in \(F_i\)). Let \(\gamma\) touch \(F_i\) at point \(Q_i\).

Now let us construct the sphere \(\omega_i\) passing through \(\Gamma_i\) and tangent to \(\gamma\) at point \(X_i\). Let \(M_i\) be the second meeting point of \(\omega_i\) and \(X_iQ_i\). Let \(Q\) and \(M\) be the convex polytopes with vertices \(Q_1, \ldots, Q_k\) and \(M_1, \ldots, M_k\), respectively. For every \(j \in \{1, \ldots, n\}\), let \(\ell_j\) be the radical plane of spheres \(\gamma\) and \(P_j\) (hence \(\ell_j\) passes through the midpoints of \(P_jQ_i\) for all \(i\) such that \(P_j \in F_i\)).

Let \(m_j\) be the plane passing through \(P_j\) perpendicular to \(IP_j\). Let \(S_i\) be the circumcenter of sphere passing through \(\Gamma_i\) and \(I\).

Now we proceed with the analogues of results from the previous section.

In the proof of Theorem 11 below we clarify how to generalize the proof of Theorem 3 to the case of a polytope. All the other proofs of Theorems below are completely analogous to those in the case of triangle; thus we omit these proofs.

**Theorem 9** (cf. Theorem 1). Line \(OM_i\) is the perpendicular to \(F_i\). Further, point \(M_i\) has equal powers with respect to \(\gamma\) and any point \(X \in \Gamma_i\) (here \(X\) is regarded as a degenerate sphere).

**Corollary 10** (cf. Corollary 2). If \(P_j \in F_i\), then \(\ell_j\) passes through \(M_i\).

**Theorem 11** (cf. Theorem 3). Points \(M_1, \ldots, M_k\) lie on some sphere \(\Omega\) with center \(O\). Next, polytopes \(\mathcal{M}\) and \(\mathcal{Q}\) are homothetical with some center \(K\).

**Proof.** To prove the first statement, it suffices to prove the equality \(OM_i = OM_s\) for every pair of indices \(i, s\) such that \(F_i\) and \(F_s\) correspond to adjacent faces.

Let \(F_{i,s}\) be the plane containing \(I\) and \(F_i \cap F_s\); then \(F_{i,s}\) is the bisector plane of \(F_i\) and \(F_s\). By Theorem 1, each of \(M_i\) and \(M_s\) has equal powers with respect to \(\gamma\) and all the points in \(\Gamma_i \cap \Gamma_s\), hence \(M_i, M_s\) is perpendicular to the plane spanned.

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\(^8\)In \(n\)-dimensional case, these two hyperfaces should have a common \((n-2)\)-dimensional face.
by $\Gamma_i \cap \Gamma_s$ and $I$ which is $F_{i,s}$. Finally, by the same Theorem we have $OM_i \perp F_i$ and $OM_s \perp F_s$, hence $\angle OM_i M_s = \angle OM_s M_i$, as required.

The proof of the second statement of Theorem is completely analogous to that in the case of triangle. \qed

Now, as in the case of triangle, we denote by $h$ the obtained homothety taking $Q$ to $M$, by $K$ its center, and by $\Omega$ the circumsphere of $M$ (then $\Omega = h(\gamma)$). Denote also by $R(\Omega)$ the radius of $\Omega$.

**Remark 9.** Notice that the first statement of Theorem 11 in the case of tetrahedron appeared as a problem proposed by F. Bakharev in All-Russian mathematical olympiad in 2003 [2, Problem 744].

**Corollary 12** (cf. Corollary 4). Lines $Q_i X_i = X_i M_i$ ($i = 1, \ldots, k$) are concurrent at point $K$.

**Corollary 13** (cf. Corollary 5). Points $K$, $I$, and $O$ are collinear, and $\frac{KO}{KI} = \frac{R(\Omega)}{r}$.

**Theorem 14** (cf. Theorem 6). $K$ has equal powers with respect to spheres $\omega_i$ ($i = 1, \ldots, k$).

**Theorem 15** (cf. Theorem 7). Let $i \in \{1, \ldots, k\}$. Consider a point $I_i$ such that $M_i$ is the midpoint of $Q_i I_i$. Then for all $j \in \{1, \ldots, n\}$ with $P_j \in F_i$, plane $m_j$ passes through $I_i$.

**Remark 10.** From Theorem 15 we see that the convex polytope $\mathcal{I}$ with vertices $I_1, \ldots, I_k$ is also homothetical to the polytopes $Q$ and $M$. A particular case of this fact (for an inscribed and circumscribed quadrilateral in the plane) appeared as a problem by S. Berlov, L. Emelyanov, and A. Smirnov in All-Russian mathematical olympiad in 2004 [2, Problem 755].

**Theorem 16** (cf. Theorem 8). All the points $S_i$ lie on the sphere $\Gamma_S$ with center $O$ and radius $R_S = R(\Omega) - r/2$.

**Remark 11.** The alternative approach mentioned in the previous section also works in the general case. It uses the following generalized statement.

**Proposition 2.** Let $\Gamma$ and $\gamma$ be two fixed spheres, and let $\Gamma' \subset \Gamma$ be a circle. Suppose two spheres $\omega_1$ and $\omega_2$ (each may degenerate to a plane) passing through $\Gamma'$ are tangent to $\gamma$ at $X$ and $Y$, respectively. Then the line $XY$ passes through a fixed point $K$ that is the pole of radical plane of $\Gamma$ and $\gamma$ with respect to $\gamma$.

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