SOME PROPERTIES OF THE BROCARD POINTS OF
A CYCLIC QUADRILATERAL

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Abstract. In this article we have constructed the Brocard points of a cyclic quadrilateral, we have found some of their properties and using these properties we have proved the problem of A. A. Zaslavsky.

1. The Problem

Alexey Zaslavsky, Brocard’s points in quadrilateral [4].

Given a convex quadrilateral $ABCD$. It is easy to prove that there exists a unique point $P$ such that $\angle PAB = \angle PBC = \angle PCD$. We will call this point Brocard point ($\text{Br}(ABCD)$) and the respective angle Brocard angle ($\phi(ABCD)$) of broken line $ABCD$. Note some properties of Brocard’s points and angles:

- $\phi(ABCD) = \phi(DCBA)$ if $ABCD$ is cyclic;
- if $ABCD$ is harmonic then $\phi(ABCD) = \phi(BCDA)$. Thus there exist two points $P,Q$ such that $\angle PAB = \angle PBC = \angle PCD = \angle PDA = \angle QBA = \angle QCB = \angle QDC = \angle QAD$. These points lie on the circle with diameter $OL$ where $O$ is the circumcenter of $ABCD$, $L$ is the common point of its diagonals and $\angle POL = \angle QOL = \phi(ABCD)$.

Problem. Let $ABCD$ be a cyclic quadrilateral, $P_1 = \text{Br}(ABCD)$, $P_2 = \text{Br}(BCDA)$, $P_3 = \text{Br}(CDAB)$, $P_4 = \text{Br}(DABC)$, $Q_1 = \text{Br}(DCBA)$, $Q_2 = \text{Br}(ADCB)$, $Q_3 = \text{Br}(BADC)$, $Q_4 = \text{Br}(CBAD)$. Then $S_{P_1P_2P_3P_4} = S_{Q_1Q_2Q_3Q_4}$.

I first saw this problem 2-3 years ago in an article by Alexei Myakishev (see [3]). I did not consider solving it then, but my pupils built a structure very similar (seemingly) to the described. Thus Zaslavsky’s problem served as a stimulus (accelerator) to develop a good article (related to isogonal conjugate points), which they presented on international events. I sincerely hope that this article will appear on the pages of “Kvant”. Now they are university students, but again I saw the problem in the “Journal of Classical Geometry”. At first I thought that using the developed article I would figure out the solution quickly, but I did not. A new construction came out.

2. Some Properties Of Brocard Points Of a Triangle.

Let $P$ and $Q$ be the first and the second Brocard points of a $\triangle ABC$. We will prove that the Brocard points and the three vertices $A$, $B$ and $C$ define three pairs of similar triangles, and three pairs of triangles with equal areas (Fig. 1).
Proposition 1. If \( P \) and \( Q \) are the first and the second Brocard points of a \( \triangle ABC \), then:

(i) \( \triangle ABP \sim \triangle CBQ \), \( \triangle BCP \sim \triangle ACQ \) and \( \triangle CAP \sim \triangle BAQ \);

(ii) \( S_{ABP} = S_{ACQ} \), \( S_{BCP} = S_{BAQ} \) and \( S_{CAP} = S_{CBQ} \).

Proof. If we denote with \( \varphi \) the Brocard angle (Fig. 2), then \( \angle BAP = \angle BCQ = \varphi \), and from the construction ([1]) of the Brocard points of \( \triangle ABC \) we have \( \angle APB = \angle CQB = 180^\circ - \angle B \). Hence \( \triangle ABP \sim \triangle CBQ \).

We construct the altitudes \( PP_c, PP_a, QQ_c \) and \( QQ_a \). Then, having \( \triangle ABP \sim \triangle CBQ \), \( \triangle P_a BP \sim \triangle Q_a BQ \) and \( \triangle Q_c BP \sim \triangle P_a BQ \), we have the following equations:

\[
\frac{S_{BPC}}{S_{BAQ}} = \frac{BC \cdot PP_a}{AB \cdot QQ_c} = \frac{QQ_a}{PP_c \cdot QQ_c} = \frac{BP}{BP} \cdot \frac{BP}{BQ} = 1.
\]

For the other pairs of triangles, the proof is analogous. \( \square \)

We are going to find, similar to these properties, for the Brocard points of a cyclic quadrilateral.

3. Construction

Let the quadrilateral \( ABCD \) be a cyclic, \( AB \cap CD = E \) and let for exactitude \( B \) is between \( A \) and \( E \), \( AD \cap BC = F \) and \( D \) is between \( A \) and \( F \). We denote \( \angle BAD = \alpha \) and \( \angle ABC = \beta \).
In order to construct the Brocard points of the quadrilateral $ABCD$, we firstly construct the points $M_1, M_2, M_3, M_4$ and $N_1, N_2, N_3, N_4$, as shown in Table 1:

|   | $M_1$ ∈ $AD$ and $BM_1$ || $CD$ | $N_1$ ∈ $AD$ and $CN_1$ || $BA$ |
|---|---|---|---|---|---|
| $M_2$ | $M_2$ ∈ $AB$ and $CM_2$ || $DA$ | $N_2$ ∈ $AB$ and $DN_2$ || $CB$ |
| $M_3$ | $M_3$ ∈ $BC$ and $DM_3$ || $AB$ | $N_3$ ∈ $BC$ and $AN_3$ || $DC$ |
| $M_4$ | $M_4$ ∈ $CD$ and $AM_4$ || $BC$ | $N_4$ ∈ $CD$ and $BN_4$ || $AD$ |

Table 1

Now we construct the intersection points of the pairs of circumcircles of the triangles, as shown in Table 2:

<table>
<thead>
<tr>
<th></th>
<th>$\triangle BAM_1$</th>
<th>$\triangle DCN_3$</th>
<th>$\triangle BCM_2$</th>
<th>$\triangle DAM_4$</th>
<th>$\triangle DCM_3$</th>
<th>$\triangle DCN_1$</th>
<th>$\triangle BAN_3$</th>
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<tbody>
<tr>
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<td>$P_4$</td>
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<tr>
<td>$\triangle DAM_4$</td>
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<td>$P_3$</td>
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</table>

Table 2

**Proposition 2.** The points $P_1, P_2, P_3, P_4$ and $Q_1, Q_2, Q_3, Q_4$ are the Brocard points of the quadrangle $ABCD$.

**Proof.** Let $Q_4$ be the intersection point of the circumcircles of triangles $BAN_3$ and $BCN_4$ (Table 2). We denote $\angle Q_4CB = \varphi_4$ (Fig. 3). Since $BN_4 || AD$ and $AN_3 || DC$ (Table 1), then $\angle CN_4B = \beta$ and $\angle BN_3A = \alpha$. So:

- $\angle ABQ_4 = \beta - \angle CBQ_4 = \beta - \angle CN_4Q_4 = \beta - (\beta - \varphi_4) = \varphi_4$,
- $\angle DAQ_4 = \alpha - \angle BAQ_4 = \alpha - (180^\circ - \angle AQ_4B - \varphi_4) = \alpha - (180^\circ - (180^\circ - \alpha) - \varphi_4) = \varphi_4$

Hence $Q_4 = Br(CBAD)$ is a Brocard point and $\varphi_4 = \phi(CBAD)$ is a Brocard angle in the quadrilateral $ABCD$. As $ABCD$ is cyclic quadrilateral, hence $\varphi_4 =$
\(\phi(DABC)\). For the other points, the proof is analogous. If we denote the Brocard angles in \(ABCD\) with \(\varphi_1, \varphi_2, \varphi_3\) and \(\varphi_4\) then we have:

\[
\begin{align*}
\varphi_1 &= \phi(ABCD) = \phi(DCBA) = \angle P_1AB = \angle P_1BC = \\
&= \angle P_1CD = \angle Q_1DC = \angle Q_1CB = \angle Q_1BA, \\
\varphi_2 &= \phi(BCDA) = \phi(ADCB) = \angle P_2BC = \angle P_2CD = \\
&= \angle P_2DA = \angle Q_2AD = \angle Q_2DC = \angle Q_2CB, \\
\varphi_3 &= \phi(CDAB) = \phi(BADC) = \angle P_3CD = \angle P_3DA = \\
&= \angle P_3AB = \angle Q_3BA = \angle Q_3AD = \angle Q_3DC, \\
\varphi_4 &= \phi(DABC) = \phi(CBAD) = \angle P_4DA = \angle P_4AB = \\
&= \angle P_4BC = \angle Q_4CB = \angle Q_4BA = \angle Q_4AD.
\end{align*}
\]

4. Some properties of the Brocard points of a cyclic quadrilateral

**Proposition 3.** The triads of points \(C, P_1, M_1; D, P_2, M_2; A, P_3, M_3; B, P_4, M_4; B, Q_1, N_1; C, Q_2, N_2; D, Q_3, N_3\) and \(A, Q_4, N_4\) are collinear.

**Proof.** \(\angle AQ_4B + \angle BQ_4N_4 = (\alpha - 180^\circ) + \alpha = 180^\circ\) (Fig. 3). For the other triads of points, the proof is analogous. \(\square\)

**Proposition 4.** The fours of lines \(CM_1, DM_2, AM_3, BM_4\) and \(BN_1, CN_2, DN_3, AN_4\) are concurrent.

![Fig. 4.](image)

**Proof.** Let \(AN_4 \cap BN_1 = Q_0\) (Fig. 4). Then using \(\triangle AQ_0N_1 \sim \triangle N_4Q_0B\), \(\triangle AED \sim \triangle BEN_4\) and \(\triangle N_1CD \sim \triangle BEN_4\) (see Table 1) we have the following equations:

\[
\frac{AQ_0}{Q_0N_4} = \frac{AN_1}{BN_4} = \frac{AD}{BN_4} - \frac{DN_1}{BN_4} = \frac{DE}{EN_4} - \frac{DC}{EN_4} = \frac{CE}{EN_4}.
\]
Now, let $AD \cap DN_3 = Q'_0$. From the similarity of $\triangle AN_3Q'_0 \sim \triangle N_4DQ'_0$, $\triangle AN_3B \sim \triangle ECB$ and also because $AD \parallel BN_4$ we have consecutively the following equations:

$$\frac{AQ'_0}{Q'_0N_4} = \frac{AN_3}{DN_4} = \frac{CE.AB}{BE.DN_4} = \frac{CE}{BE} \cdot \frac{BE}{EN_4} = \frac{CE}{EN_4}.$$ 

Then $\frac{AQ_0}{Q_0N_4} = \frac{AQ'_0}{Q'_0N_4}$ and $Q_0 \equiv Q'_0$. In the same way $Q_0 \in CN_2$. So the lines $BN_1, CN_2, DN_3, AN_4$ intersect in a point $Q_0$, and similarly the lines $CM_1, DM_2, AM_3, BM_4$ have a common point $P_0$.

Because $\triangle BEN_4 \sim \triangle CEB$ and the Law of Sines used for $\triangle CEB$, we attain:

$$\frac{AQ_0}{Q_0N_4} = \frac{CE}{EN_4} = \frac{CE^2}{BE^2} = \frac{\sin^2 \beta}{\sin^2 \alpha}.$$ 

In the same way we have the following equations:

$$\frac{AQ_0}{Q_0N_4} = \frac{CQ_0}{Q_0N_2} = \frac{AP_0}{P_0M_3} = \frac{CP_0}{P_0M_1} = \frac{\sin^2 \beta}{\sin^2 \alpha},$$

$$\frac{BQ_0}{Q_0N_1} = \frac{DQ_0}{Q_0N_3} = \frac{BP_0}{P_0M_4} = \frac{DP_0}{P_0M_2} = \frac{\sin^2 \alpha}{\sin^2 \beta}.$$ 

Now we will show that the points $P_0, Q_0$ and the vertices $A, B, C$ and $D$ define four pairs of triangles with equal areas (Fig. 5).

Proposition 5. If $P_0$ and $Q_0$ are the intersection points of the fours of lines $CM_1, DM_2, AM_3, BM_4$ and $BN_1, CN_2, DN_3, AN_4$ (see Proposition 4), then:

$$S_{ABP_0} = S_{DAQ_0}, S_{BCP_0} = S_{ABQ_0}, S_{CDP_0} = S_{BCQ_0} \text{ and } S_{DAP_0} = S_{CDQ_0}.$$
Taking into account that the fours of points $B$, $P_0$, $P_4$, $M_4$ and $A$, $Q_0$, $Q_4$, $N_4$ (Fig. 6) are collinear (Proposition 3 and Proposition 4) and equations (2) we attain:

\begin{align}
S_{BCP_0} &= \frac{BP_0}{BM_4} S_{BCM_4} = \frac{\sin^2 \alpha}{\sin^2 \alpha + \sin^2 \beta} S_{BCM_4}, \\
S_{ABQ_0} &= \frac{AQ_0}{AN_4} S_{BN_4A} = \frac{\sin^2 \beta}{\sin^2 \alpha + \sin^2 \beta} S_{BN_4A},
\end{align}

Since $\angle BCM_4 = \angle ABN_4 = 180^\circ - \alpha$ (see Table 1) and $\angle CBM_4 = \angle BN_4A = \varphi_4$ (see (1)), then $\triangle BCM_4 \sim \triangle BN_4A$ and

\begin{align}
\frac{S_{BCM_4}}{S_{BN_4A}} &= \frac{BC^2}{BN_4^2} = \frac{\sin^2 \beta}{\sin^2 \alpha} \quad \text{(Law of Sines for $\triangle CBN_4$)}
\end{align}

Using equations (3) and (4) we calculate

\begin{align}
\frac{S_{BCP_0}}{S_{ABQ_0}} &= \frac{\sin^2 \alpha}{\sin^2 \beta} \cdot \frac{S_{BCM_4}}{S_{BN_4A}} = 1.
\end{align}

The proof, of the areas equality, of the other triangle pairs is similar.

Now we can consider the quadrangles $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$. We will prove that the points $P_0$, $Q_0$ and the vertices of quadrangles $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$ define four pairs of triangles with equal areas (Fig. 7).
Proposition 6. If $P_0$ and $Q_0$ are the intersection points of the fours of lines $CM_1$, $DM_2$, $AM_3$, $BM_4$ and $BN_1$, $CN_2$, $DN_3$, $AN_4$ (see Proposition 4), then:

$$S_{P_1P_2P_0} = S_{Q_1Q_2Q_0}, \quad S_{P_2P_3P_0} = S_{Q_2Q_3Q_0},$$
$$S_{P_3P_4P_0} = S_{Q_3Q_4Q_0}, \quad S_{P_4P_1P_0} = S_{Q_4Q_1Q_0}.$$  

Proof. Firstly we will show that the areas of the pairs of triangles $CDP_1$, $BCQ_1$ and $CDP_2$, $BCQ_2$ are equal (Fig. 8). The proof is similar to that of Proposition 1. Since $\angle P_1BC = \angle Q_1DC = \varphi_1$ and $\angle P_1CB = \angle Q_1CD$ (see equations (1)), hence $\triangle BCP_1 \sim \triangle DCQ_1$. We construct the altitudes $P_1P_b$, $P_1P_c$, $Q_1Q_b$ and $Q_1Q_c$.

Then using $\triangle BCP_1 \sim \triangle DCQ_1$, $\triangle P_1P_bC \sim \triangle Q_1Q_cC$ and $\triangle P_1P_cC \sim \triangle Q_1Q_bC$, we have the following equations:

$$\frac{S_{CDP_1}}{S_{BCQ_1}} = \frac{DC}{BC} \frac{P_1P_c}{Q_1Q_c} = \frac{Q_1Q_c}{P_1P_b} \frac{P_1P_c}{Q_1Q_b} = \frac{Q_1C}{P_1C} \frac{P_1C}{Q_1C} = 1.$$  

So we attain: $S_{CDP_1} = S_{BCQ_1}$ and $S_{CDP_2} = S_{BCQ_2}$ (analogous).
Since the triads of points $C, P_1, P_0; \ D, P_0, P_2; \ B, Q_1, Q_0$ and $C, Q_0, Q_2$ are collinear (Propositions 3 and 4) and $S_{CDP_0} = S_{BCQ_0}$ (Proposition 5), hence

$$S_{CP_0P_2} = S_{BQ_0Q_2} \text{ and } S_{DP_0P_1} = S_{CQ_0Q_1}.$$

We calculate $S_{P_1P_2P_0} = \frac{P_0P_1}{P_0C} \cdot S_{CR_0P_2} = \frac{S_{DP_0P_1}}{S_{CDP_0}} \cdot S_{CR_0P_2}$ and

$$S_{Q_1Q_2Q_0} = \frac{Q_0Q_1}{Q_0B} \cdot S_{BQ_0Q_2} = \frac{S_{CQ_0Q_1}}{S_{BCQ_0}} \cdot S_{BQ_0Q_2}, \text{ therefore } S_{P_1P_2P_0} = S_{Q_1Q_2Q_0}.$$

The proof, of the areas equality, of the other pairs of triangles is similar. \hfill \Box

From Proposition 6 immediately follows

**Consequence** $S_{P_1P_2P_3P_4} = S_{Q_1Q_2Q_3Q_4}$.

We should note that another solution is given by Chandan Banerjee in his blog [2].

5. Two Additional Properties

Above we have proved that $\triangle BCP_1 \sim \triangle DCQ_1$ and $S_{CDP_1} = S_{BCQ_1}$. In fact, we have also $\triangle ABP_1 \sim \triangle CBQ_1$ and $S_{ABP_1} = S_{ABQ_1}$. In general, each pair of Brocard points $P_i$ and $Q_i$, in a cyclic quadrilateral $ABCD$ and its corresponding vertices, define two pairs of triangles with equal areas (to compare see Proposition 1 and Fig. 1).
On the other hand, the points $P_0$ and $Q_0$ not only “stole” the pairs of triangles with equal areas related to vertices of the quadrilateral $ABCD$ (see Proposition 5 and Fig. 5), but they have and those in quadrilaterals $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$ (see Proposition 6 and Fig. 7). Naturally arise the question, where are the pairs of similar triangles associated with $P_0$, $Q_0$ and vertices of quadrilaterals $ABCD$, $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$. It turns out that they appear in a very curious way.

In Fig. 9 we show that the triangles defined by the point $P_0$ and quadrilateral $P_1P_2P_3P_4$ are similar to the triangles determined by a point $Q_0$ and quadrilateral $ABCD$. Conversely, the triangles defined by the point $Q_0$ and quadrilateral $Q_1Q_2Q_3Q_4$ are similar to the triangles determined by a point $P_0$ and quadrilateral $ABCD$ (Fig. 9).

**Proposition 7.** If $P_0$ and $Q_0$ are the intersection points of the fours of lines $CM_1$, $DM_2$, $AM_3$, $BM_4$ and $BN_1$, $CN_2$, $DN_3$, $AN_4$ (see Proposition 4), then:

\[
\begin{align*}
\triangle P_1P_2P_3 & \sim \triangle CBQ_0, & \triangle P_2P_3P_0 & \sim \triangle DCQ_0, \\
\triangle P_3P_4P_0 & \sim \triangle ADQ_0, & \triangle P_4P_1P_0 & \sim \triangle BAQ_0
\end{align*}
\]

and

\[
\begin{align*}
\triangle Q_1Q_2Q_0 & \sim \triangle DCP_0, & \triangle Q_2Q_3Q_0 & \sim \triangle ADP_0, \\
\triangle Q_3Q_4Q_0 & \sim \triangle BAP_0, & \triangle Q_4Q_1Q_0 & \sim \triangle CBP_0.
\end{align*}
\]

**Proof.** We will prove that $\triangle Q_1Q_2Q_0 \sim \triangle DCP_0$ (Fig. 8). The points $Q_1$ and $Q_2$ lie on the circumcircle of the triangle $\triangle DCN_1$ (Table 2) therefore $\angle Q_1Q_2C = \angle Q_1DC = \varphi_4$. But $\angle Q_2Q_0Q_1 = \angle Q_0CB + \angle Q_1BC = \beta - \varphi_1 + \varphi_2$ (see equations (1)) and $\angle CP_0D = 180^\circ - (\angle P_0DC + \angle P_1CD) = 180^\circ - (180^\circ - \beta - \varphi_2 + \varphi_1) = \beta - \varphi_1 + \varphi_2$, namely $\angle Q_2Q_0Q_1 = \angle CP_0D$. Hence $\triangle Q_1Q_2Q_0 \sim \triangle DCP_0$. The similarity of the other pairs of triangles we prove similarly.

From Proposition 7 follows, that $\angle P_1 + \angle P_3 = \angle Q_2 + \angle Q_4 = \angle A + \angle C = 180^\circ$ (see Fig. 9), which means that quadrilaterals $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$ are cyclic.

6. APPENDIX

I apply the problems of my students.

Anton Belev, Kaloyan Bucovsky,

**Problem 1.** If in the quadrangle $ABCD$ is inscribed ellipse with foci $F_1$ and $F_2$, and the isogonal conjugated points of $F_1$ and $F_2$ with respect to $\triangle DAB$, $\triangle ABC$, $\triangle BCD$, $\triangle CDA$ are $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$, then the quadrangles $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are congruent.

**Problem 2.** In the quadrangle $ABCD$ is inscribed a circle with center $O$. If the isogonal conjugated points of $O$ with respect to $\triangle DAB$, $\triangle ABC$, $\triangle BCD$, $\triangle CDA$ are $A_1B_1C_1$ and $D_1$, then $A_1B_1C_1D_1$ is a parallelogram. If $AC \perp BD$, then $A_1B_1C_1D_1$ is a rhombus. If around the quadrangle $ABCD$ is described a circle, then $A_1B_1C_1D_1$ is a rectangle.
The construction, which reveals how quadrangles $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ merged into a parallelogram (when $F_1$ get near to $F_2$) is really fascinating. We did it using GeoGebra.

REFERENCES


