Abstract. Let $A_1, B_1, C_1$ be points chosen on the sidelines $BC, CA, BA$ of a triangle $ABC$, respectively. The circumcircles of triangles $AB_1C_1, BC_1A_1, CA_1B_1$ intersect the circumcircle of triangle $ABC$ again at points $A_2, B_2, C_2$ respectively. We prove that triangle $A_2B_2C_2$ is similar to triangle $A_3B_3C_3$, where $A_3, B_3, C_3$ are symmetric to $A_1, B_1, C_1$ with respect to the midpoints of the sides $BC, CA, BA$ respectively.

**Theorem 1.** Let $A_1, B_1, C_1$ be points chosen on the sidelines $BC, CA, BA$ of a triangle $ABC$, respectively. The circumcircles of triangles $AB_1C_1, BC_1A_1, CA_1B_1$ intersect the circumcircle of triangle $ABC$ again at points $A_2, B_2, C_2$ respectively. Points $A_3, B_3, C_3$ are symmetric to $A_1, B_1, C_1$ with respect to the midpoints of the sides $BC, CA, BA$ respectively. Then the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.
Preliminary. Let us introduce some notions and formulate known lemmas that we use in the proof.

We will work with oriented angles between lines. For two straight lines $\ell, m$ in the plane, $\angle(\ell, m)$ denotes the angle of counterclockwise rotation which transform line $\ell$ into a line parallel to $m$ (the choice of the rotation centre is irrelevant). This is signed quantity; values differing by a multiple of $\pi$ are identified, so that $\angle(\ell, m) = -\angle(m, \ell)$, $\angle(\ell, m) + \angle(m, n) = \angle(\ell, n)$.

If $\ell$ is the line through the points $K, L$ and $m$ is the line through $M, N$, one writes $\angle(KL, MN)$ for $\angle(\ell, m)$; the characters $K, L$ are freely interchangeable; and so are $M, N$. The counterpart of the classical theorem about cyclic quadrilaterals is the following:

Lemma 1. Four non-collinear points $K, L, M, N$ are concyclic if and only if

$$\angle(KM, LM) = \angle(KN, LN).$$

Further we use (1) without explicit reference.

Lemma 2. Suppose that $A_1, B_1, C_1$ are points on the sidelines $BC, CA, BA$ of a triangle $ABC$, respectively; then the three circles $(AB_1C_1)$, $(BC_1A_1)$, $(CA_1B_1)$ have a common point.

Proof. Let $(AB_1C_1)$ and $(BC_1A_1)$ intersect at $C_1$ and $P$. Then

$$\angle(PA_1, CA_1) = \angle(PA_1, BA_1) = \angle(PC_1, BC_1) = \angle(PC_1, AC_1) = \angle(PB_1, AB_1) = \angle(PB_1, CB_1).$$

The equality between the outer terms shows that the points $A_1, B_1, P, C$ are concyclic. Thus $P$ is the common point of the three mentioned circles. □
Lemma 3. Let $A_1, B_1, C_1$ be points on the sidelines $BC, CA, BA$ of a triangle $ABC$, respectively; and the circles $(AB_1C_1), (BC_1A_1), (CA_1B_1)$ meet at $P$. Suppose that the lines $AP, PB, CP$ meet the circle $(ABC)$ again at $A', B', C'$, respectively; then triangles $A_1B_1C_1$ and $A'B'C'$ are similar. (In particular, the pedal triangle of $P$ is similar to $A'B'C'$.)

Proof. We have

\[ \angle(A_1B_1, C_1B_1) = \angle(A_1B_1, PB_1) + \angle(PB_1, C_1B_1) = \angle(A_1C, PC) + \angle(PA, C_1A). \]

On the other hand,

\[ \angle(A'B', C'B') = \angle(A'B', BB') + \angle(BB', C'B') = \angle(AA', BA) + \angle(BC, C'C). \]

But the lines $A'A, BA, BC, C'C$ coincide respectively with $PA, C_1A, A_1C, PC$. So the sums on the right-hand of (2) and (3) are equal, that leads to $\angle(A_1B_1, C_1B_1) = \angle(A'B', C'B')$. Hence (by cyclic shift, once more) also

\[ \angle(B_1C_1, A_1C_1) = \angle(B'C', A'C') \text{ and } \angle(C_1A_1, B_1A_1) = \angle(C'A', B'A'). \]

This means that triangles $A_1B_1C_1$ and $A'B'C'$ are similar. \qed

Proof of the Theorem. Let the circles $(AB_1C_1), (BC_1A_1), (CA_1B_1)$ meet at $P$ (see Lemma 2), and let

\[ \varphi = \angle(PA_1, BC) = \angle(PB_1, CA) = \angle(PC_1, AB). \]
Let lines \( A_2P, B_2P, C_2P \) meet the circle \((ABC)\) again at \( A_4, B_4, C_4 \), respectively. Since

\[
\angle(A_4A_2, AA_2) = \angle(PA_2, AA_2) = \angle(PC_1, AC_1) = \angle(PC_1, AB) = \varphi,
\]

we have \( \angle(OA_4, OA) = 2\varphi \) (here \( O \) is the center of \((ABC)\)). Hence \( A \) is the image of \( A_4 \) under rotation by \( 2\varphi \) about \( O \). The same rotation takes \( B_4 \) to \( B \), and \( C_4 \) to \( C \). Thus triangle \( ABC \) is the image of \( A_4B_4C_4 \) under this rotation, therefore

\[
\angle(A_4B_4, AB) = \angle(B_4C_4, BC) = \angle(C_4A_4, CA) = 2\varphi.
\]

Further, we have \( \angle(AB_4, AB) = \angle(B_2B_4, B_2B) = \varphi \). Hence by (4)

\[
\angle(AB_4, PC_1) = \angle(AB_4, AB) + \angle(AB, PC_1) = \varphi + (-\varphi) = 0,
\]

which means that \( AB_4 \parallel PC_1 \).

Let \( C_5 \) be the intersection of lines \( PC_1 \) and \( A_4B_4 \); define \( A_5, B_5 \) analogously. So \( AB_4 \parallel C_1C_5 \) and, by (5) and (4),

\[
\angle(A_4B_4, PC_1) = \angle(A_4B_4, AB) + \angle(AB, PC_1) = 2\varphi + (-\varphi) = \varphi;
\]

i.e., \( \angle(B_4C_5, C_5C_1) = \varphi \). This combined with \( \angle(C_5C_1, C_1A) = \angle(PC_1, AB) = \varphi \) (see (4)) proves that the quadrilateral \( AB_4C_5C_1 \) is an isosceles trapezoid with \( AC_1 = B_4C_5 \).
Suppose $\overrightarrow{AC_3} = \lambda \overrightarrow{AB}$; then $\overrightarrow{BC_1} = \lambda \overrightarrow{BA}$, and $\overrightarrow{A_4C_5} = \lambda \overrightarrow{A_4B_4}$. In other words, the rotation which maps triangle $A_4B_4C_4$ onto $ABC$ carries $C_5$ onto $C_3$. Likewise, it takes $A_5$ to $A_3$, and $B_5$ to $B_3$. So the triangles $A_3B_3C_3$ and $A_5B_5C_5$ are congruent.

Lines $B_4C_5$ and $PC_5$ coincide respectively with $A_4B_4$ and $PC_1$. Thus by (6)

$$\angle(B_4C_5, PC_5) = \varphi.$$ 

Analogously (by cyclic shift) $\varphi = \angle(C_4A_5, PA_5)$, which rewrites as

$$\varphi = \angle(B_4A_5, PA_5).$$
These relations imply that the points $P, B_4, C_5, A_5$ are concyclic. Analogously $P, C_4, A_5, B_5$ and $P, A_4, B_5, C_5$ are concyclic quadruples.

Now it is sufficient to apply Lemma 3 for triangle $A_4B_4C_4$ and points $A_5, B_5, C_5$. It provides similarity of triangles $A_2B_2C_2$ and $A_5B_5C_5$. This ends the proof of Theorem. \hfill \Box

References


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